Points

September 27, 2016

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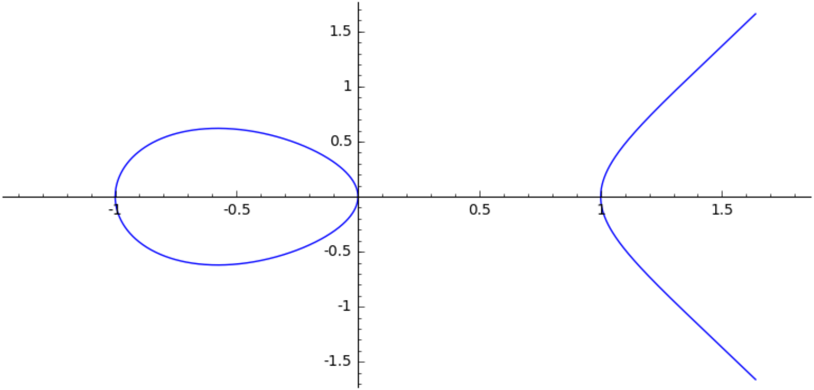

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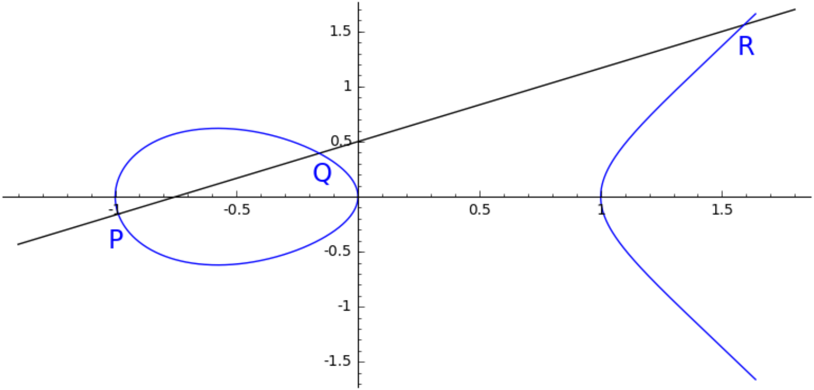

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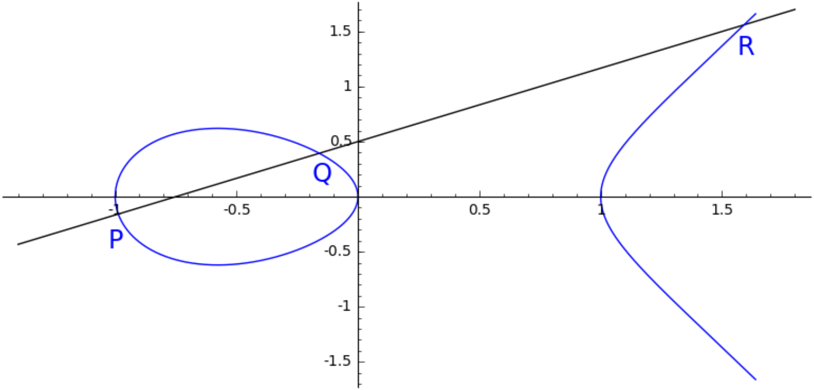

$$P + Q + R = Id_E$$

Question: how many points are in E ?

Question: how many points are in $E(\mathbb{F}_{3^d})$?

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- $\alpha + \beta \in \mathbb{Z}$ and $|\alpha + \beta| \leq 2\sqrt{3}$

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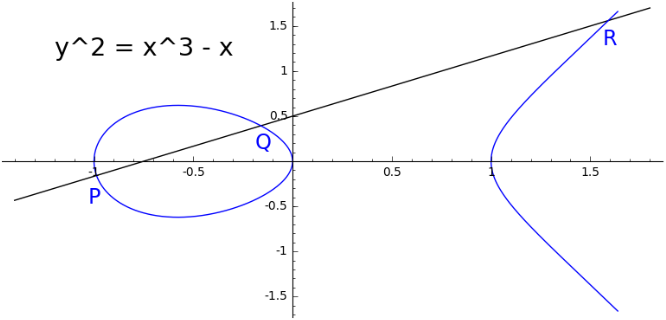
$E[2]$ and $P(t) = t^2 + at \pm 3$ $\quad -3 \leq a \leq 3$



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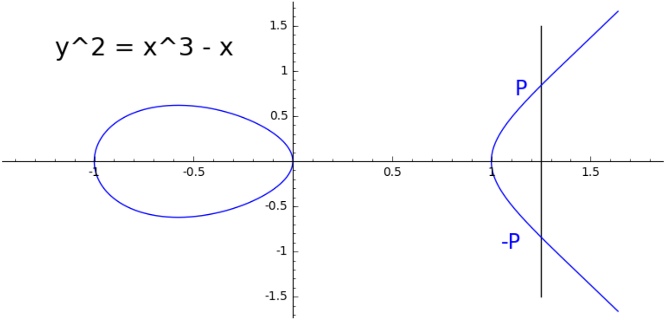

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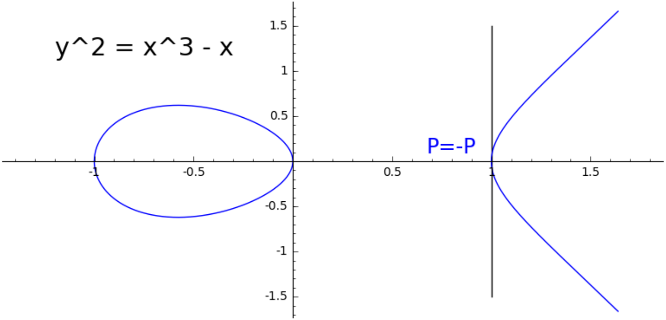

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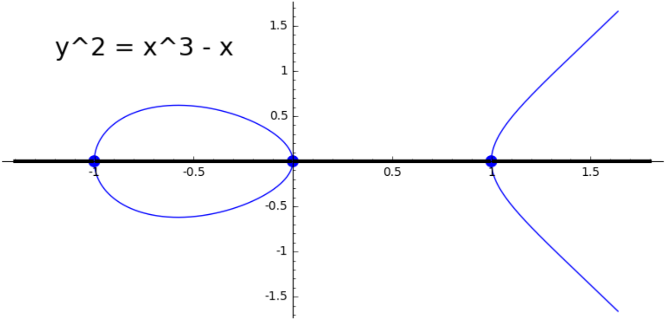

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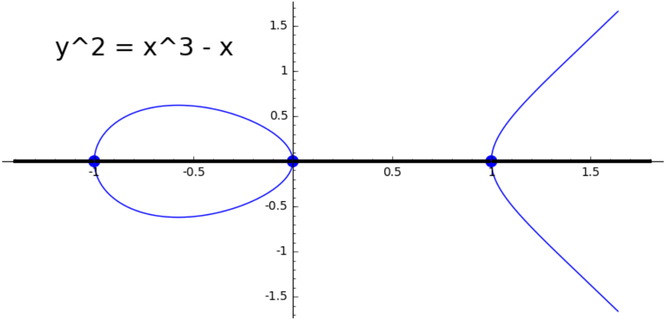

- $E[2] = \ (0,0),\ (1,0),\ (-1,0),\ \overset{Id_E}{\underset{\infty}{||}}$

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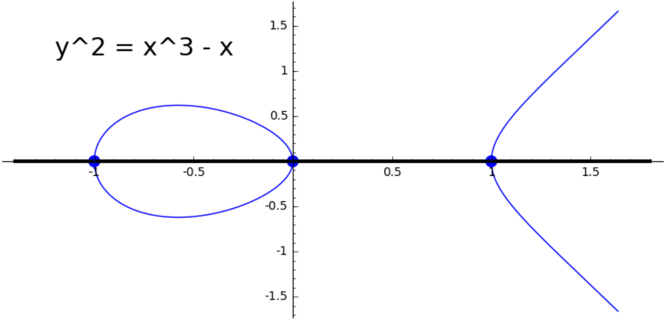

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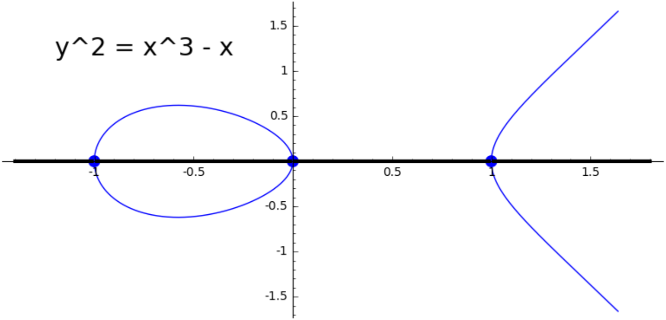

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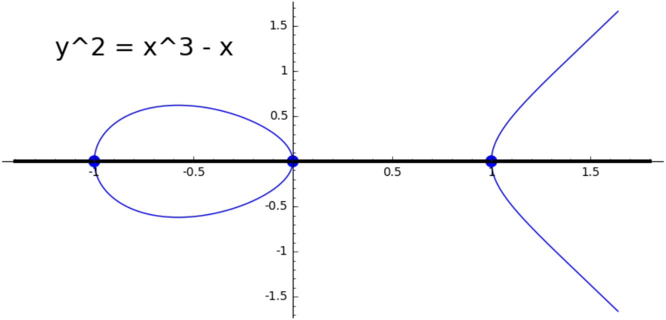

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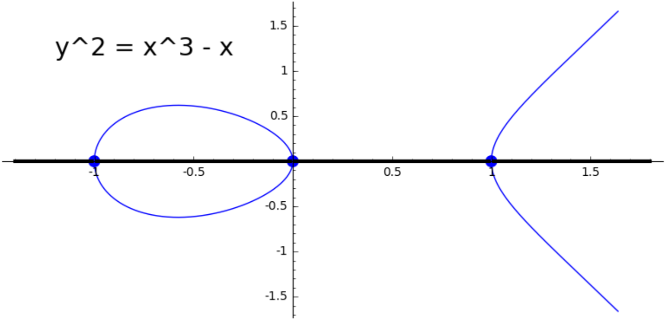

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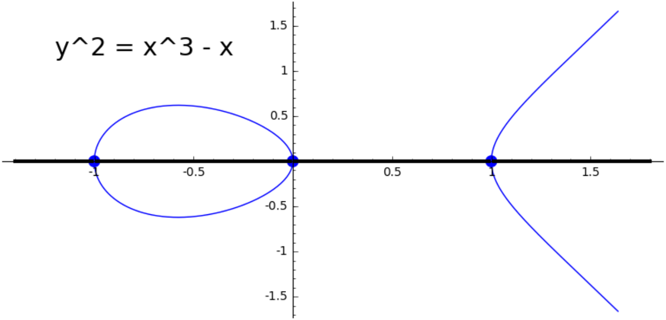

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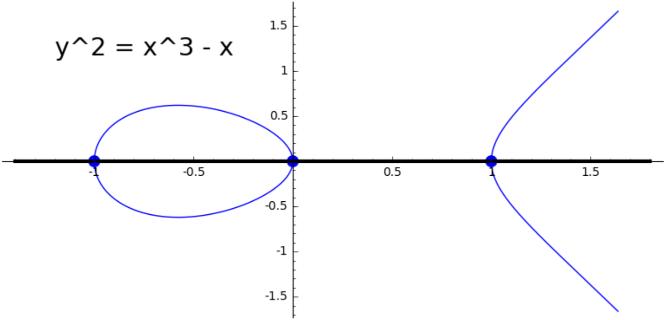

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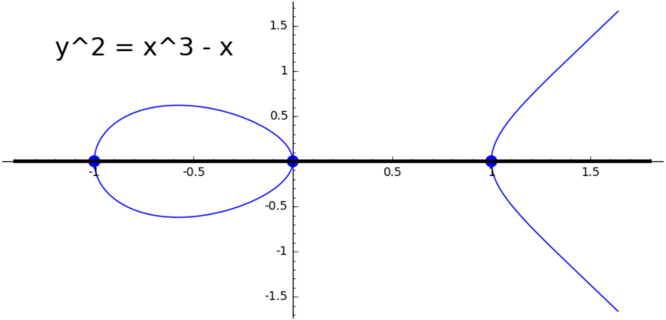

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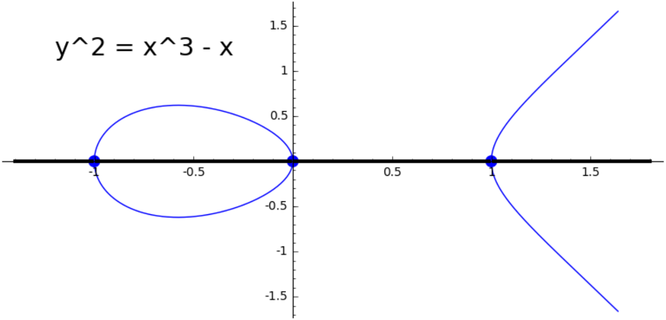

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- $char(Frob_{E[2]}) = t^2 + 1 \qquad \in (\mathbb{Z}/2\mathbb{Z})[t]$
- $\Rightarrow P(t) \equiv t^2 + 1$ modulo 2

$E[4]$

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$+$	$0e_1$	$1e_1$	$2e_1$	$3e_1$
$0e_2$	[0 : 1 : 0]	(i , i–1)	(0,0)	(i , -i+1)
$1e_2$	(i+1, i–1)	(i–1, -i+1)	(-i+1, -i–1)	(-i–1, -i–1)
$2e_2$	(1,0)	(-i , i+1)	(-1,0)	(-i , -i–1)
$3e_2$	(i+1, -i+1)	(-i–1, i+1)	(-i+1, i+1)	(i–1, i–1)

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$0e_2$	$[0:1:0]$	(i , i–1)	(0,0)	(i , -i+1)
$1e_2$	(i+1, i–1)	(i–1, -i+1)	(-i+1, -i–1)	(-i–1, -i–1)
$2e_2$	(1,0)	(-i , i+1)	(-1,0)	(-i , -i–1)
$3e_2$	(i+1, -i+1)	(-i–1, i+1)	(-i+1, i+1)	(i–1, i–1)

$Frob(e_1)$

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$Frob(e_1) \quad =$ (i^3, $\mathrm{i}^3 - 1^3$)

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$3e_2$	(i+1, -i+1)	(-i–1, i+1)	(-i+1, i+1)	(i–1, i–1)

$$\begin{aligned} Frob(e_1) &= (\ i^3,\ i^3 - 1^3\ ) \\ &= (\ \text{-}i,\ \text{-}i - 1\ ) \end{aligned}$$

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$1e_2$	$(i+1, i-1)$	$(i-1, -i+1)$	$(-i+1, -i-1)$	$(-i-1, -i-1)$
$2e_2$	$(1,0)$	$(-i, i+1)$	$(-1,0)$	$(-i, -i-1)$
$3e_2$	$(i+1, -i+1)$	$(-i-1, i+1)$	$(-i+1, i+1)$	$(i-1, i-1)$

$$\begin{aligned} Frob(e_1) &= (i^3, i^3 - 1^3) \\ &= (-i, -i-1) \qquad = 3e_1 + 2e_2 \end{aligned}$$

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$0e_2$	$[0 : 1 : 0]$	$(\mathrm{i}, \mathrm{i}-1)$	$(0,0)$	$(\mathrm{i}, -\mathrm{i}+1)$
$1e_2$	$(\mathrm{i}+1, \mathrm{i}-1)$	$(\mathrm{i}-1, -\mathrm{i}+1)$	$(-\mathrm{i}+1, -\mathrm{i}-1)$	$(-\mathrm{i}-1, -\mathrm{i}-1)$
$2e_2$	$(1,0)$	$(-\mathrm{i}, \mathrm{i}+1)$	$(-1,0)$	$(-\mathrm{i}, -\mathrm{i}-1)$
$3e_2$	$(\mathrm{i}+1, -\mathrm{i}+1)$	$(-\mathrm{i}-1, \mathrm{i}+1)$	$(-\mathrm{i}+1, \mathrm{i}+1)$	$(\mathrm{i}-1, \mathrm{i}-1)$

$$\begin{aligned}
Frob(e_1) &= (\mathrm{i}^3, \mathrm{i}^3 - 1^3) \\
&= (-\mathrm{i}, -\mathrm{i} - 1) \qquad = 3\, e_1 + 2\, e_2 \\
Frob(e_2) &= (\mathrm{i}^3 + 1, \mathrm{i}^3 - 1^3) \\
&= (-\mathrm{i}+1, -\mathrm{i} - 1) \qquad = 2\, e_1 + 1\, e_2
\end{aligned}$$

E[4] and $P(t) = t^2 + at \pm 3$

$$\Rightarrow Frob_{E[4]} \qquad = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad \in GL_2(\mathbb{Z}/4\mathbb{Z})$$

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$$\Rightarrow P(t) \equiv t^2 + 3 \quad \text{modulo } 4$$

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\end{aligned}$$

Recall that Theorem $\Rightarrow P(t) = t^2 + ax \pm 3, \quad -3 \leq a \leq 3$

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$\#E(\mathbb{F}_{3^d})$

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d	$1^d - \alpha^d - (-\alpha)^d + 3^d$	$\#E(\mathbb{F}_{3^d})$

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d	$1^d - \alpha^d - (-\alpha)^d + 3^d$	$\#E(\mathbb{F}_{3^d})$
1	$1 - \alpha + \alpha + 3$	4

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where α, β *are the roots of*

$$P(t) = t^2 + 3.$$

$$\alpha = \sqrt{-3}, \qquad \beta = -\sqrt{-3} = -\alpha$$

d	$1^d - \alpha^d - (-\alpha)^d + 3^d$	$\#E(\mathbb{F}_{3^d})$
1	$1 - \alpha + \alpha + 3$	4
2	$1 - (-3) - (-3) + 9$	

$\#E(\mathbb{F}_{3^d})$

Theorem $\Rightarrow$

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1	$1 - \alpha + \alpha + 3$	4
2	$1 - (-3) - (-3) + 9$	16

$+$	$0e_1$	$1e_1$	$2e_1$	$3e_1$
$0e_2$	[0 : 1 : 0]	(i , i–1)	(0,0)	(i , -i+1)
$1e_2$	(i+1, i–1)	(i–1, -i+1)	(-i+1, -i–1)	(-i–1, -i–1)
$2e_2$	(1,0)	(-i , i+1)	(-1,0)	(-i , -i–1)
$3e_2$	(i+1, -i+1)	(-i–1, i+1)	(-i+1, i+1)	(i–1, i–1)

$\#E(\mathbb{F}_{3^d})$

Theorem

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Theorem

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d	$1^d - \alpha^d - (-\alpha)^d + 3^d$	$\#E(\mathbb{F}_{3^d})$
1	$1 - \alpha + \alpha + 3$	4
2	$1 - (-3) - (-3) + 9$	16
3	$1 - \alpha^3 + \alpha^3 + 27$	28