Universal approach to β-matrix models

M.Shcherbina

Institute for Low Temperature Physics, Ukr. Ac. Sci.
Kharkov, Ukraine

Princeton, 01.04.2014

Model definition

Distributions in $\mathbb{R}^n$, depending on the function V and $\beta > 0$

$$p_{n,\beta}(\lambda_1, ..., \lambda_n) = Z_n^{-1}[\beta, V]e^{\beta H(\lambda_1,...,\lambda_n)/2},$$

where H (Hamiltonian) and $Z_n[\beta, V]$ (partition function) are

$$H(\lambda_1, \ldots, \lambda_n) = -n\sum_{i=1}^{n} V(\lambda_i) + \sum_{i\neq j} \log|\lambda_i - \lambda_j|,$$

$$Z_n[\beta, V] = \int e^{\beta H(\lambda_1,...,\lambda_n)/2} d\lambda_1 \ldots d\lambda_n,$$

$$V(\lambda) > (1 + \varepsilon)\log(1 + \lambda^2).$$

For $\beta = 1, 2, 4$ it is a joint eigenvalues distribution of real symmetric, hermitian and symplectic matrix models respectively.

Expectation and correlation functions

$$\text{For given } h : \mathbb{R}^n \to \mathbb{C}, \quad \langle h \rangle_{V,n} = \int h(\lambda_1, \dots, \lambda_n) p_{n,\beta}(\lambda_1, ..., \lambda_n) d\bar{\lambda}$$

Correlation functions (marginal densities):

$$p^{(m)}_{n,\beta}(\lambda_1, ..., \lambda_m) = \int_{\mathbb{R}^{n-l}} p_{n,\beta}(\lambda_1, ...\lambda_m, \lambda_{m+1}, ..., \lambda_n) d\lambda_{m+1} ... d\lambda_n$$

The linear eigenvalue statistics (LES) and the counting measure of eigenvalues

$$\mathcal{N}_n[h] = \sum_{j=1}^{n} h(\lambda_j), \quad N_n[\Delta] = \sum_{j=1}^{n} 1_\Delta(\lambda_j).$$

Main problems of the global regime

1. weak limit of the first correlation function $\mathrm{w}-\lim_{n\to\infty} p^{(1)}_{n,\beta}(\lambda) = \rho(\lambda)$, support σ of $\rho(\lambda)$;
2. weak limits of the other correlation functions $p^{(m)}_{n,\beta}(\lambda_1, ..., \lambda_m)$ and their factorization property

$$p^{(m)}_{n,\beta}(\lambda_1, ..., \lambda_m) - p^{(1)}_{n,\beta}(\lambda_1)...p^{(1)}_{n,\beta}(\lambda_m) \to 0, \quad \text{as} \quad n \to \infty;$$

3. large deviation type bounds for the correlation functions;
4. generating functional of LES

$$\Phi[t,h] = \langle e^{\beta t \mathcal{N}_n[h]/2}\rangle_{V,n} = \frac{Z_n[\beta, V - \frac{1}{n}h]}{Z_n[\beta, V]}$$

   and CLT for LES.
5. expansion in n^{-1} for $\log Z_n[\beta, V]$ and correlation functions;

Main problems of the local regime

1. Universality of local eigenvalue statistics. In the bulk case it means that for any $\lambda_0 \in \sigma$ $(\rho(\lambda_0) \neq 0)$ all correlation functions after a proper scaling have limits which do not depend on V, i.e. the limits

$$\lim_{n\to\infty} (\rho(\lambda_0))^{-m} p^{(m)}_{n,\beta}(\lambda_0 + s_1/n\rho(\lambda_0), \dots, \lambda_0 + s_m/n\rho(\lambda_0))$$

coincide with that for the Gaussian case $V^*(\lambda) = \frac{1}{2}\lambda^2$.

2. Universality of gap probabilities. For a fixed system of nonintersecting intervals $\bar\Delta = (\Delta_1, \dots, \Delta_k)$ and $\bar m = (m_1, \dots, m_k)$ introduce the indicators functions

$$\Psi_{\bar\Delta,\bar m}(\bar\lambda; \lambda_0) := 1_{N_n(\lambda_0 + \frac{\Delta_1}{n\rho(\lambda_0)}) = m_1, \dots, N_n(\lambda_0 + \frac{\Delta_k}{n\rho(\lambda_0)}) = m_k}.$$

Universality means that $\lim_{n\to\infty} \langle \Psi_{\bar\Delta,\bar m}(\bar\lambda; \lambda_0)\rangle_{V,n} = \lim_{n\to\infty} \langle \Psi_{\bar\Delta,\bar m}(\bar\lambda; 0)\rangle_{*,n}$

3. Universality of the generating functional, which has the form

$$\Psi_\phi(\bar\lambda; \lambda_0) := \prod_{j=1}^{n} \Big(1 - \phi\big(n\rho(\lambda_0)(\lambda_j - \lambda_0)\big)\Big), \quad 0 \le \phi(x) \le 1, \quad |\mathrm{supp}\,\phi| < \infty.$$

The equilibrium problem

$$\mathcal{E}[\mathrm{V}] = -\min_{\mathrm{m}\in\mathcal{M}_1}\Big\{-\mathrm{L}[\mathrm{dm},\mathrm{dm}] + \int \mathrm{V}(\lambda)\mathrm{m}(\mathrm{d}\lambda)\Big\} = \mathcal{E}_{\mathrm{V}}(\mathrm{m}^*),$$

$$\text{where}\quad \mathrm{L}[\,\mathrm{dm},\mathrm{dm}'] = \int \log|\lambda-\mu|\mathrm{dm}(\lambda)\mathrm{dm}'(\mu),$$

For any continuous V the problem has a unique solution m^*. If V' is a Hölder function then $\mathrm{m}^*(\mathrm{d}\lambda)$ has the density $\mathrm{m}^*(\mathrm{d}\lambda) = \rho(\lambda)\mathrm{d}\lambda$ with a compact support $\sigma := \operatorname{supp}\mathrm{m}^*$. The density ρ is an equilibrium density and it is uniquely defined by the condition

$$\mathrm{v}(\lambda) := 2\int \log|\lambda-\mu|\rho(\mu)\mathrm{d}\mu - \mathrm{V}(\lambda) = \mathrm{v}^* = \mathrm{const},\quad \lambda\in\sigma$$

$$\mathrm{v}(\lambda) \le \mathrm{v}*,\quad \lambda\notin\sigma$$

Without loss of generality we can assume that $\mathrm{v}^* = 0$.

The first step for the global regime

Theorem [Boutet de Monvel, Pastur, S:95; Johansson:98]

If V is a Hölder function, then

$$\log Z_n[\beta, V] = \frac{n^2\beta}{2}\mathcal{E}[V] + O(n\log n),$$

where $\mathcal{E}[V] = \mathcal{E}_V(m^*)$.

Moreover, if $h' \in L_2[\sigma_\varepsilon]$

$$|n^{-1}E\{\mathcal{N}_n[h]\} - (h, m^*)| \le Cn^{-1/2}\log^{1/2} n||h'||_2^{1/2}||h||_2^{1/2}$$

Small perturbations for one cut potentials

Theorem [Johansson:98]

V is a polynomial, $\sigma = [-2, 2]$, and ρ is "generic", $h : \mathbb{R} \to \mathbb{R}$ with $||h^{(6)}||_\infty, ||h'||_\infty \le \epsilon n^{1/3}$, $\dot{h} := h - (\rho, h)$

$$\langle e^{\beta \mathcal{N}_n[h]/2} \rangle_{V,n} = \exp\left\{ \left(1 - \frac{\beta}{2}\right)(h, \nu) + \frac{\beta}{8}\left(\overline{D}_\sigma h, h\right) \right\} \left(1 + n^{-1} O\left(||h^{(4)}||_\infty^3\right)\right)$$

where the "variance operator" $\overline{D}_\sigma$ depends only of σ, and the measure ν have the form

$$(h, \nu) := \frac{1}{4}(h(-2) + h(2)) - \frac{1}{2\pi} \int_\sigma \frac{h(\lambda) d\lambda}{\sqrt{4 - \lambda^2}} + \frac{1}{2}(D_\sigma \log P, h)$$

P is defined by the relation $\rho(\lambda) = (2\pi)^{-1} P(\lambda) \sqrt{4 - \lambda^2}$

Remark

D_σ is a rank one perturbation of $-\mathcal{L}_\sigma^{-1}$, where $\mathcal{L}_\sigma$ is the integral operator defined by the kernel $\log |\lambda - \mu|$ for the interval σ

Large deviation type bounds

Take any n-independent small $\varepsilon > 0$. It was proven in [Albeverio, Pastur, S:01] that if we replace in the definition of the partition function and of the correlation functions the integration over $\mathbb{R}$ by the integration σ_ε, then $p_{n,\beta}^{(m)}$ and the new marginal densities $p_{n,\beta}^{(m,\varepsilon)}$ for $m = 1, 2, \ldots$ satisfy the inequalities

$$\sup_{\lambda_1,\ldots,\lambda_m \in \sigma_\varepsilon} |p_{n,\beta}^{(m)}(\lambda_1, \ldots, \lambda_m) - p_{k,\beta}^{(m,\varepsilon)}(\lambda_1, \ldots, \lambda_m)| \le C_m e^{-n\beta d_\varepsilon},$$

$$Z_n[\beta, V] = Z_n^{(\varepsilon)}[\beta, V]\big(1 + e^{-n\beta d_\varepsilon}\big).$$

It is more convenient to consider the integration with respect to σ_ε, thus, starting from this moment it is assumed that this truncation is made, and below the integration without limits means the integration over σ_ε, but the superindex ε will be omitted.

Change of variables in the one cut case

Let V be some smooth enough potential with equilibrium density ρ such that $\mathrm{supp}\rho = [-2, 2]$, and $\zeta(\lambda) : \sigma_\varepsilon = [-2-\varepsilon, 2+\varepsilon] \to \sigma_\varepsilon$ be some smooth function such that $\inf_{\sigma_\varepsilon} \zeta' > 0$.
Consider

$$\mathrm{H}^{(\zeta)}(\lambda_1, \dots, \lambda_n) = -n \sum V(\zeta(\lambda_j)) + \sum_{i \neq j} \log |\zeta(\lambda_i) - \zeta(\lambda_j)| + \frac{2}{\beta} \sum \log \zeta'(\lambda_j)$$

It is evident that the corresponding partition function and all the marginal densities satisfy the relations

$$\begin{aligned} Z_{n,\beta}^{(\zeta)} &:= \int e^{\beta \mathrm{H}^{(\zeta)}/2} d\bar{\lambda} = Z_n[\beta, V] \\ p_{n,\beta}^{(m,\zeta)}(\lambda_1, \dots, \lambda_m) &:= (Z_{n,\beta}^{(\zeta)})^{-1} \int e^{\beta \mathrm{H}^{(\zeta)}/2} d\lambda_{m+1} \dots d\lambda_n \\ &= p_{n,\beta}^{(m)}(\zeta(\lambda_1), \dots, \zeta(\lambda_m)) \end{aligned}$$

On the other hand,

$$H^{(\zeta)}(\lambda_1,\dots,\lambda_n) = -\,n\sum V(\zeta(\lambda_j)) + \sum_{i\neq j}\log|\lambda_i-\lambda_j|$$
$$+\sum_{i,j}\log\left|\frac{\zeta(\lambda_i)-\zeta(\lambda_j)}{\lambda_i-\lambda_j}\right| + (\frac{2}{\beta}-1)\sum\log\zeta'(\lambda_j)$$

Denote

$$L^{(\zeta)}(\lambda,\mu) := \log\left|\frac{\zeta(\lambda)-\zeta(\mu)}{\lambda-\mu}\right| = L_+^{(\zeta)}(\lambda,\mu) - L_-^{(\zeta)}(\lambda,\mu) = \sum \eta_k\psi_k(\lambda)\psi_k(\mu),$$

where $L_+^{(\zeta)}$ and $L_-^{(\zeta)}$ are positive compact operators in $L_2[\mathbb{R}]$ having smooth kernels (there is some freedom here which we will be used below).

For sufficiently smooth $\zeta(\lambda)$ these operators have smooth eigenfunctions $\{\psi_{k\pm}(\lambda)\}_{k=1}^{\infty}$ and eigenvalues $\{\eta_{k\pm}\}_{k=1}^{\infty}$ such that if we denote $\psi_{2k-1}(\lambda) := \psi_{k+}(\lambda)$, $\psi_{2k}(\lambda) := \psi_{k-}(\lambda)$ and $\eta_{2k-1} = \eta_{k+}$, $\eta_{2k} = \eta_{k-}$ the convergence above is uniform in σ_ε

Choice of $\zeta(\lambda)$

Choose $\zeta(\lambda)$ from the equation

$$\zeta'(\lambda) = \frac{\rho_{\mathrm{sc}}(\lambda)}{\rho(\zeta(\lambda))}, \quad \zeta(-2) = -2,$$

where

$$\rho_{\mathrm{sc}}(\lambda) = (2\pi)^{-1}\sqrt{4-\lambda^2},$$

and

$$\rho(\lambda) = (2\pi)^{-1}\mathrm{P}(\lambda)\sqrt{4-\lambda^2}$$

is the equilibrium density corresponding to V.

Then

$$\zeta(2) = 2 \qquad \rho(\zeta(\lambda))\zeta'(\lambda) = \rho_{\mathrm{sc}}(\lambda)$$

and $\zeta(\lambda)$ could be extended to σ_ε with the same number of derivatives as P.

For this choice of ζ write

$$\sum_{i,j} L^{(\zeta)}(\lambda_i,\lambda_j) = \sum_k \eta_k \Big(\sum_j \psi_k(\lambda_i)\Big)^2 = \sum_k \eta_k \Big(\sum_j (\psi_k(\lambda_j) - (\psi_k,\rho_{sc})\Big)^2$$

$$+ 2n \sum_j \sum_k \eta_k \psi_k(\lambda_j)(\psi_k,\rho_{sc}) - n^2 \sum_k \eta_k (\psi_k,\rho_{sc})^2$$

$$= R(\bar\lambda) + 2n \sum_j \int L^{(\zeta)}(\lambda_j,\mu)\rho_{sc}(\mu)d\mu - n^2 \int L^{(\zeta)}(\lambda,\mu)\rho_{sc}(\lambda)\rho_{sc}(\mu)d\lambda d\mu$$

where $(f,g) := \int fgd\lambda$. It is easy to see that

$$2\int L^{(\zeta)}(\lambda_j,\mu)\rho_{sc}(\mu)d\mu = V(\zeta(\lambda_j)) - \frac{\lambda_j^2}{2},$$

$$\int L^{(\zeta)}(\lambda,\mu)\rho_{sc}(\lambda)\rho_{sc}(\mu)d\lambda d\mu = \mathcal{E}_{sc} - \mathcal{E}_V =: -\Delta\mathcal{E}.$$

Hence we finally obtain that our Hamiltonian has the form:

$$H^{(\zeta)}(\bar\lambda) = -n\sum \frac{\lambda_j^2}{2} + \sum_{i\neq j} \log|\lambda_i - \lambda_j| + \Big(\frac{2}{\beta} - 1\Big)\sum \log \zeta'(\lambda_j) + R(\bar\lambda) + n^2\Delta\mathcal{E}$$

Linearization of $R(\bar\lambda)$

Consider the Hamiltonian

$$H_n(\bar\lambda) = H_n^*(\bar\lambda) + (1-\frac{2}{\beta})\sum \log\zeta'(\lambda_j) + \frac{1}{2}\sum_{k=1}^{M}\eta_k\Big(\sum_j(\psi(\lambda_j)-(\psi_k,\rho_{sc}))\Big)^2,$$

where H_n^* is the Hamiltonian corresponding to $V^*(\lambda)=\lambda^2/2$. Write for any $1\le k\le M$

$$\exp\Big\{\frac{\beta}{2}\eta_k\Big(\sum_j(\psi(\lambda_j)-(\psi_k,\rho_{sc}))\Big)^2\Big\}$$
$$=\sqrt{\frac{\beta}{8\pi}}\int\exp\Big\{\frac{\beta}{2}\Big(\sqrt{\eta_k}\Big(\sum_j(\psi_k(\lambda_j)-(\psi_k,\rho_{sc}))\Big)u_k-u_k^2/4\Big)\Big\}$$

and denote

$$h_{\bar u}(\lambda)=\sum_{k=1}^{M}\sqrt{\eta_k}\psi_k(\lambda)u_k+(\frac{2}{\beta}-1)\log\zeta'(\lambda),\quad \dot h_{\bar u}=h_{\bar u}-(h_{\bar u},\rho_{sc}).$$

Global regime

We obtain

$$\frac{Z_n[V,\beta]}{Z_n[V^*,\beta]} = \exp\{\beta n^2 \Delta\mathcal{E}/2 + n(1-\frac{\beta}{2})(\log\zeta',\rho_{sc})\}$$
$$\cdot\left(\frac{\beta}{8\pi}\right)^{M/2}\int e^{-\beta(\bar u,\bar u)/8}\langle e^{\beta\mathcal{N}_n[\dot h_u]/2}\rangle_{*,n}d\bar u$$

Then for $\langle e^{\beta\mathcal{N}_n[\dot h_u]/2}\rangle_{*,n}$ the Johansson theorem yields

$$Z_n[V,\beta] = Z_n[V^*,\beta]\exp\left\{\frac{\beta}{2}n^2\Delta\mathcal{E} + n(1-\frac{\beta}{2})(\log\zeta',\rho_{sc})\right\}$$
$$\cdot\left(\frac{\beta}{8\pi}\right)^{M/2}\int \exp\{-\frac{\beta}{8}(\bar u,\bar u)+\frac{\beta}{8}(\overline{D}_\sigma h_{\bar u},h_{\bar u})\}(1+o(1)O((u,u)^2))d\bar u.$$

The only fact which we need to prove is that the integral with respect to $\bar u$ is convergent.

Local bulk regime.

To study the gap probabilities we consider $\Psi_{\bar{\Delta},\bar{m}}(\bar{\lambda};\lambda_0)$.
After the change of variables we obtain that $\Psi_{\bar{\Delta},\bar{m}}(\bar{\lambda};\lambda_0)$ will be transform into the indicator function $\Psi^{(\zeta)}_{\bar{\Delta},\bar{m}}(\bar{\lambda};\zeta^{-1}(\lambda_0))$ of the same type but for the new system of intervals: each interval $\Delta_j=(a_j,b_j)$, $j=1,\dots,k$ or

$$\lambda_0+a_j/n\rho(\lambda_0)\le\lambda\le\lambda_0+b_j/n\rho(\lambda_0)$$

should be replaced by

$$\begin{aligned}\lambda_0+a_j/n\rho(\lambda_0)\le\zeta(\lambda)\le\lambda_0+b_j/n\rho(\lambda_0)\\ \Leftrightarrow\zeta^{-1}(\lambda_0+a_j/n\rho(\lambda_0))\le\lambda\le\zeta^{-1}(\lambda_0+b_j/n\rho(\lambda_0)).\end{aligned}$$

But, e.g., for the left edge point we have

$$\begin{aligned}\zeta^{-1}(\lambda_0+a_j/n\rho(\lambda_0))=&\zeta^{-1}(\lambda_0)+a_j/n\rho(\lambda_0)\zeta'(\lambda_0)+O(n^{-2})\\ =&\zeta^{-1}(\lambda_0)+a_j/n\rho_{sc}(\zeta^{-1}(\lambda_0))+O(n^{-2})\end{aligned}$$

Hence we indeed have the indicator function of the same type.

To prove the universality of correlation functions in the weak form, it suffices to take arbitrary smooth functions $\phi_j(x)$ $(j = 1, \dots k)$ and to consider the limits of the expectations of the functions of the form

$$\Phi_k(\bar{\lambda}; \lambda_0) = \prod_{j=1}^{k} \Big(n^{-1} \sum_{i=1}^{n} n\phi_j\big(n\rho(\lambda_0)(\lambda_i - \lambda_0)\big)\Big), \ \lambda_0 \in (-2 + \varepsilon, 2 - \varepsilon),$$

we need to replace $\Phi_k(\bar{\lambda}; \lambda_0)$ by

$$\Phi_k^{(\zeta)} = \prod_{j=1}^{k} \Big(\sum_{i} \varphi_j(n\rho(\lambda_0)(\zeta(\lambda_i) - \lambda_0))\Big),$$

Then in the case of the indicator functions we get

$$\langle \Psi_{\bar{\Delta},\bar{m}} \rangle_{V,n} = \langle \Psi^{(\zeta)}_{\bar{\Delta},\bar{m}} \rangle_{H^{(\zeta)}}$$
$$= I_n^{-1} \Big(\frac{\beta}{8\pi}\Big)^{M/2} \int e^{-\beta(\bar{u},\bar{u})/8} d\bar{u} \langle \Psi^{(\zeta)}_{\bar{\Delta},\bar{m}} e^{\beta \mathcal{N}_n[\dot{h}_u]/2} \rangle_{*,n} = \langle \Psi^{(\zeta)}_{\bar{\Delta},\bar{m}} \rangle_{*,n}$$
$$+ I_n^{-1} \Big(\frac{\beta}{8\pi}\Big)^{M/2} \int e^{-\beta(\bar{u},\bar{u})/8} d\bar{u} \Big(\langle \Psi^{(\zeta)}_{\bar{\Delta},\bar{m}} e^{\beta \mathcal{N}_n[\dot{h}_u]/2} \rangle_{*,n} - \langle \Psi^{(\zeta)}_{\bar{\Delta},\bar{m}} \rangle_{*,n} \langle e^{\beta \mathcal{N}_n[\dot{h}_u]/2} \rangle_{*,n} \Big)$$

where I_n is the normalization constant. It is $e^{O(1)}$, so can give only some additional constant in the bounds.

The first step: real h

Lemma

Let $V_h = V^* + \frac{1}{n}h$ with real analytic h such that $||h^{(3)}||_2 \le C \log n$ Then

$$\left| \frac{\langle \Psi^{(\zeta)}_{\bar{\Delta},\bar{m}} e^{\beta \mathcal{N}_n[\dot{h}]/2} \rangle_{*,n}}{\langle e^{\beta \mathcal{N}_n[\dot{h}]/2} \rangle_{*,n}} - \langle \Psi^{(\zeta)}_{\bar{\Delta},\bar{m}} \rangle_{*,n} \right| \le \varepsilon_n \to 0$$

Apply the change of variables procedure to $V_h = V^* + \frac{1}{n}h$. But then h should be a "good" perturbation, i.e. the equilibrium density, corresponding to V_h, should have the support $[-2, 2]$. Hence, one should find a, b such that the function

$$\tilde{h}(\lambda) = h(\lambda) - \ell(\lambda), \quad \ell(\lambda) := a\lambda^2 - b\lambda,$$

is a "good" perturbation ($a = (h', f_a)$, $b = (h', f_b)$ with some fixed f_a, f_b), and apply the change of variables to $V_{\tilde{h}}$.

Since

$$\zeta_h(\lambda) = \lambda + n^{-1}\tilde{\zeta}_h(\lambda) \qquad ,$$

the corresponding integral operator kernel will be

$$\log\left(1 + \frac{1}{n}\frac{\zeta_h(\lambda) - \zeta_h(\mu)}{\lambda - \mu}\right) = \frac{1}{n}\tilde{\mathcal{L}}_h(\lambda, \mu)$$

Then, completing the change of variables, we obtain that it suffices to check that

$$\langle \Psi^{(\zeta)}_{\bar{\Delta},\bar{m}} (e^{\beta R(\bar{\lambda})/2n} - 1) \rangle_{*,n} \to 0$$

It is easy, since

$$\langle (e^{\beta R(\bar{\lambda})/2n} - 1)^2 \rangle_{*,n} \to 0.$$

To remove ℓ we use the following result

Corollary from the result of Valko and Virag (09)

$$|\langle \Psi_{\bar{\Delta},\bar{m}} (\bar{\lambda}, \lambda_0 + t/n) \rangle_{*,n} - \langle \Psi_{\bar{\Delta},\bar{m}} (\bar{\lambda}, 0) \rangle_{*,n}| \le \varepsilon_n \to 0, \quad n \to \infty,$$

where the first bound is uniform for $\lambda_0 \in [-2+\varepsilon, 2-\varepsilon]$, and the second relation is uniform in the same λ_0 and $|t| \le n^{1-\delta}$ if $\delta > 0$ is fixed.

Since there is no result similar to the above for the convergence of correlation functions in the case of $\Phi_k(\bar{\lambda}, \lambda_0)$ we obtain that it coincides in the limit with $\Phi_k(\bar{\lambda}, \lambda_0 + t(h)/n)$ where $t(h) = (h', f)$ with some smooth f depending on V.

The second step: complex h

Lemma

Let the analytic in $t \in \mathcal{D} = \{t : |t| \le \log^{1/2} \varepsilon_n^{-1}, \Im t \ge 0\}$ functions F_n satisfy two bounds:

$$|F_n(t)| \le C_1 \varepsilon_n e^{t^2/2}, \quad -\log^{1/2} \varepsilon_n^{-1} \le t \le \log^{1/2} \varepsilon_n^{-1}, \quad \varepsilon_n < 1,$$
$$|F_n(t)| \le C_2 e^{(\Re t)^2/2}, \quad t \in \mathcal{D}.$$

Then the inequality

$$|F_n(t)| \le C \varepsilon_n^{1/2} |e^{t^2/2}|$$

holds for $t \in \mathcal{D}' := \frac{1}{6}\mathcal{D}$ with $C = C_1^{3/4} C_2^{1/4}$.

Main results for the one-cut case

Theorem 1 [S:13]

Let V be a smooth (possessing 7 derivatives) one-cut potential with $\sigma = [-2, 2]$ of generic behavior, and $\lambda_0 \in [-2+\varepsilon, 2-\varepsilon]$ with any fixed $\varepsilon > 0$. Then the following relations hold uniformly in $\lambda_0 \in [-2+\varepsilon, 2-\varepsilon]$:
(i) for any fixed nonintersecting intervals $\bar{\Delta}$, any fixed $\bar{m} \in \mathbb{N}^k$

$$\lim_{n\to\infty} \langle \Psi_{\bar{\Delta},\bar{m}}(\bar{\lambda}, \lambda_0)\rangle_{V,n} = \lim_{n\to\infty} \langle \Psi_{\bar{\Delta},\bar{m}}(\bar{\lambda}, 0)\rangle_{*,n}$$

(ii) any $\Psi_\phi(\bar{\lambda}, \lambda_0)$ with compactly supported piece-wise continuous ϕ

$$\lim_{n\to\infty} \langle \Psi_\phi(\bar{\lambda}, \lambda_0)\rangle_{V,n} = \lim_{n\to\infty} \langle \Psi_\phi(\bar{\lambda}, 0)\rangle_{*,n}.$$

(iii) There exists $s_* > 0$ depending on V, β and λ_0 such that for any $k \geq 1$ and any $\Phi_k(\bar{\lambda}, \lambda_0)$ with compactly supported smooth (belonging to C_1) $\{\phi_j\}_{j=1}^k$ we have

$$\lim_{n\to\infty} \left| \langle \Phi_k(\bar{\lambda}, \lambda_0)\rangle_{V,n} - \sqrt{\frac{s_*}{2\pi}} \int dt e^{-s_* t^2/2} \langle \Phi_k(\bar{\lambda} + n^{-1}t, \zeta^{-1}(\lambda_0))\rangle_{*,n} \right| = 0.$$

Multi-cut potentials and local edge regime

Theorem 2 [S:13]

Let V be a real analytic multi-cut potential with $\sigma = \cup_{\alpha=1}^{q} \sigma_\alpha$ ($\sigma_\alpha = [a_\alpha, b_\alpha]$) of generic behavior. Then, for any $\lambda_0 \in \cup_{\alpha=1}^{q} [-a_\alpha + \varepsilon, b_\alpha - \varepsilon]$ the assertions (i)–(iii) holds.

For the local edge regime the procedure is the same, but one should consider the function $\tilde{\Psi}_{\bar{\Delta},\bar{m}}(\bar{\lambda}, b_\alpha)$ which is the indicator of the set, where

$$N_n(b_\alpha + \Delta_1/n^{2/3}\gamma) = m_1, \dots N_n(b_\alpha + \Delta_k/n^{2/3}\gamma) = m_k$$

Theorem 3 [S:14]

Let V be a real analytic multi-cut potential with $\sigma = \cup_{\alpha=1}^{q} \sigma_\alpha$ ($\sigma_\alpha = [a_\alpha, b_\alpha]$)

$$\lim_{n\to\infty} \langle \tilde{\Psi}_{\bar{\Delta},\bar{m}}(\bar{\lambda}, b_\alpha) \rangle_{V,n} = \lim_{n\to\infty} \langle \tilde{\Psi}_{\bar{\Delta},\bar{m}}(\bar{\lambda}, 2) \rangle_{*,n}$$

Previous results

These results should be compared with

Theorem [Bourgade, Erdos, Yau: 11-13]

If V is a one-cut potential of generic behavior and $|V^{(4)}| \le C$, then for any k, and any smooth φ_j with a compact support

$$\lim_{n\to\infty}(2n^{\alpha-1})^{-1}\int_{-n^{-1+\alpha}}^{n^{-1+\alpha}} dt\Big(\langle\Phi_k(\lambda_0+t)\rangle_{V,n}-\langle\Phi_k(\lambda_0+t)\rangle_{*,n}\Big)=0$$

Similar results were obtained for the edge universality.

Problems

(1) potentials with "hard edges"

(2) potentials with non generic behavior of the equilibrium density ("double scaling" case, etc.)