# Golden Gates in $PU_n(\mathbb{R})$ and the Density Hypothesis

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# Golden Gates

A finite subset $S$ of a compact Lie group $L$, is called golden gate if:

## Optimal (Topological) Generators

Almost any $y \in L$ is contained in $B_V(x)$, the ball of volume $V$ centred at $x \in S^{(\ell)}$- the set of words of $S$ of length $\ell$, where

$$V = |S^{(\ell)}|^{-1} \cdot \text{poly} \log |S^{(\ell)}|.$$

and

## Approximation (Heuristic) Algorithm

There is a polynomial algorithm, such that given $y \in L$ and $V > 0$, it outputs a word $x \in S^{(\ell)} \bigcap B_V(y)$, where

$$V = |S^{(\ell)}|^{-\dim(L)}.$$

# Lubotzky, Phillips and Sarnak

In the 80's LPS proved the following results:

**Optimal Generators**

Explicit constructions of optimal generators of $L = PU_2(\mathbb{R})$.
($\Longrightarrow$ an optimal covering of the $2$-dimensional sphere).

and

**Ramanujan Graphs**

Explicit constructions of Ramanujan regular graphs.
(Ramanujan $=$ spectrally optimal graphs).

In recent years the notions appearing in the works of LPS have seen the following generalizations for higher dimensions:

**Golden Gates**

Instead of optimal generators for $L = PU_2(\mathbb{R})$,
Golden gates for general compact Lie groups $L$.

and

**Ramanujan Complexes**

Instead of Ramanujan regular graphs,
Ramanujan irregular graphs and simplicial complexes.

# Number Theory

The proof of LPS relies on the following two results:

**Ramanujan Conjecture**

The Ramanujan conjecture for $PGU_2/\mathbb{Q}$, which follows from Deligne's Theorem and Jacquet-Langlands correspondence.

and

**Class Number One**

A $p$-arithmetic group that acts simply transitive on the Bruhat-Tits tree, which follows from Jacobi four squares Theorem.

# Problems

There are two main obstacles in extending the method of proof of LPS to higher dimensions:

**Naive Ramanujan Conjecture**

The naive Ramanujan conjecture for higher rank groups is false.

and

**Class Number One**

There are only finitely many $p$-arithmetic groups that acts simply transitive on the Bruhat-Tits buildings.

# Solutions

To overcome the failure of the NRC, we follow Sarnak's strategy from the 90's, of proving and using a density hypothesis:

**Density Hypothesis - Definition**

Let $\mathcal{F}$ be a collection of automorphic representations.
Let $\mathcal{F}(T)$ the finite subset of $\mathcal{F}$ of analytic conductor $\leq T$.
Let $\mathcal{F}(T,\sigma)$ the subset of $\mathcal{F}(T)$ of decay of matrix coefficient $\geq \sigma$.
Then the density hypothesis is the claim that for any $\epsilon > 0$,

$$|\mathcal{F}(T,\sigma)| \ll_\epsilon |\mathcal{F}(T)|^{2/\sigma+\epsilon}.$$

**Density Hypothesis - Theorem**

Let $G/\mathbb{Q}$ be a classical group such that $G(\mathbb{R})$ is compact.
Let $\mathcal{F} = L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^K$, for a compact adelic subgroup $K$.
Then $\mathcal{F}$ satisfy the density hypothesis.

# Results

Using the Density hypothesis we can prove the following results:

**Golden Gates**

Explicit constructions of golden gates for $PU_n(\mathbb{R})$, for any $n \leq 8$.

and

**Optimal Covering of Hecke Points**

Let $G/\mathbb{Q}$ be a classical group such that $G(\mathbb{R})$ is compact.
Let $X$ be a homogeneous space of the Lie group $L = G(\mathbb{R})$.
Then the $p^\ell$-Hecke points covers $X$ almost optimally.

THANK YOU
FOR LISTENING