Variational formulas for directed polymer and percolation models

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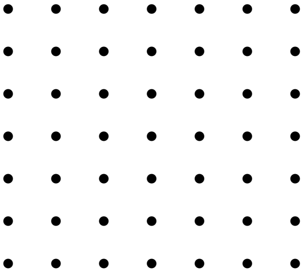
Collaborators: Nicos Georgiou, Firas Rassoul-Agha, Atilla Yilmaz

Directed polymer and percolation on $\mathbb{Z}^2$



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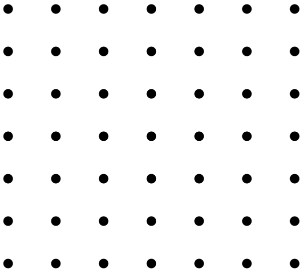

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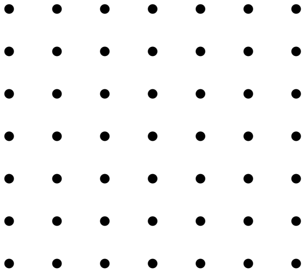

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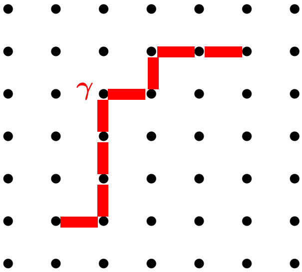

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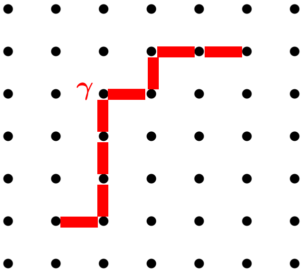

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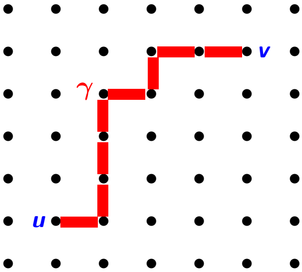

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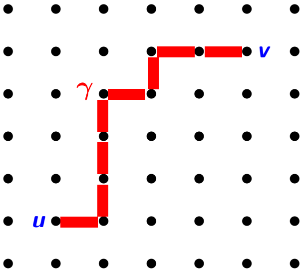

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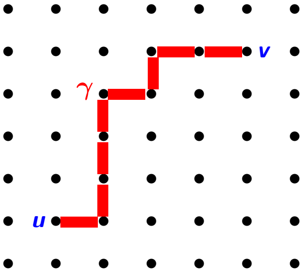

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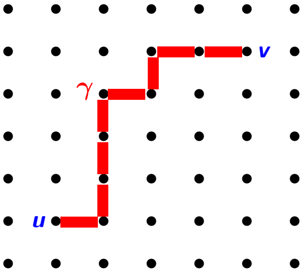

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$$g_{\mathrm{pp}}^{\beta}(\xi) \; = \; \inf_{h\in\mathbb{R}^2} \{ \, g_{\mathrm{pl}}^{\beta}(h) \, - \, \xi \cdot h \, \}$$

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This is a **large deviations duality** because (take $\beta = 1$)

$$\lim_{n\to\infty} \; n^{-1}\log Q_{(n)}\{X_n = \lfloor n\xi \rfloor\} \;=\; g_{\text{pp}}(\xi) - g_{\text{pl}}(0)$$

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Formulas for $g^{\beta}_{\text{pl}}(h)$ **and** $g^{\beta}_{\text{pp}}(\xi)$ **?**

Warm-up: periodic environment, p2ℓ case

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c	a	b	c	a	b	c	a	b
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translations $(T_x \omega)_y = \omega_{x+y}, \quad x, y \in \mathbb{Z}^2$,

act transitively.

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$$\longrightarrow \log\rho(A) \quad \text{where } \rho(A) \;=\; \text{Perron-Frobenius e-value of } A.$$

Limiting free energy for periodic environment

P2ℓ limiting free energy $\quad g_{\rm pl} = \log \rho(A) \quad$ for

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Infimum over gradients, achieved at right e-vector $\varphi = e^f$.

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Correctors $=$ L^1 closure of gradients $F(\omega, x, y) = f(T_x \omega) - f(T_y \omega)$.

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Theorem (Rassoul-Agha, S, Yılmaz, 2013)

$$g_{\mathrm{pl}}^{\beta}(h) = \inf_{\boldsymbol{F}} \ \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \ \beta^{-1}\log \sum_{i=1,2} \tfrac{1}{2}\, e^{\beta\omega_0 + \beta h\cdot e_i + \beta \boldsymbol{F}(\omega,0,e_i)}$$

where infimum over stationary L^1 correctors $\boldsymbol{F}$. A minimizing corrector exists.

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where infimum over stationary L^1 correctors $\boldsymbol{F}$. A minimizing corrector exists.

Proof Develop a quenched large deviation theory for the empirical measure $n^{-1}\sum_{k=0}^{n-1}\delta_{T_{X_k}\omega,\,X_{k+1}-X_k}$ under a fixed ω, X_k is RW.

Point-to-line last-passage percolation

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A.s. limit

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where the infimum is over stationary L^1 correctors $\boldsymbol{F}$. A minimizing corrector exists.

Periodic LPP and max-plus eigenvalue

$$g_{\mathrm{pl}}^{\infty} = \lim_{n\to\infty} n^{-1} \max_{(x_k)_{k=0}^{n-1}} \sum_{k=0}^{n-1} \omega_{x_k}$$

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$\exists\,!$ max-plus e-value $\boldsymbol{\lambda}$ and a (not unique) e-vector σ such that

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This leads to $g_{\mathrm{pl}}^{\infty} = \boldsymbol{\lambda}$ and e-value equation is

$$g_{\mathrm{pl}}^{\infty} = \max_{i=1,2} \left[\omega_0 + \sigma(T_{e_i}\omega) - \sigma(\omega) \right]$$

so variational formula for g_{pl}^{∞} minimized by $\sigma(T_{e_1}\omega) - \sigma(\omega)$.

Variational formulas for FPP and LPP

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Ours is probabilistic as explained:

- variational formula for polymers via large deviation theory (though we borrow technical ideas from homogenization work of Kosygina and Varadhan)
- zero-temperature limit.

So we have these variational formulas...

Directed polymer:

$$g_{\text{pl}}^{\beta}(h) \;=\; \lim_{n\to\infty}\; n^{-1}\,\log \sum_{(x_k)_{k=0}^{n}} 2^{-n}\,\exp\Big\{\sum_{k=0}^{n-1}\omega_{x_k} \;+\; h\cdot x_n\Big\}$$

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Switch to exactly solvable models where we can understand what goes on in these formulas.

Exactly solvable models

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- Connections with combinatorics through Robinson-Schensted-Knuth correspondence.

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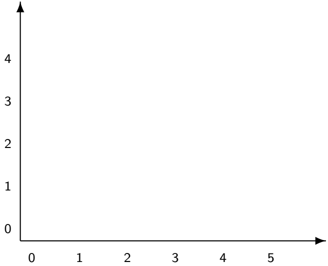
- Connections with combinatorics through Robinson-Schensted-Knuth correspondence.
- **Tractable stationary version** of the $Z_{\boldsymbol{u},\boldsymbol{v}}$ process.

Stationary log-gamma polymer



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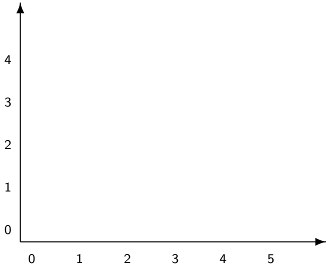

Two parameters $0 < \alpha < \rho$.

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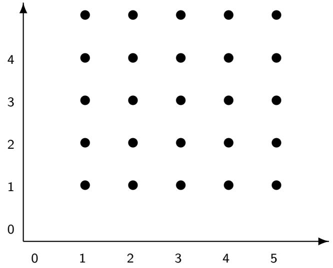

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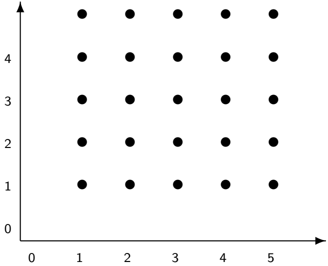

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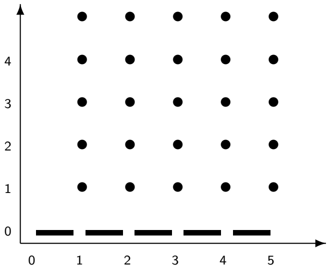

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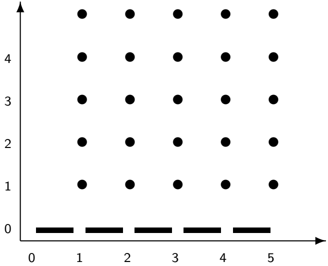

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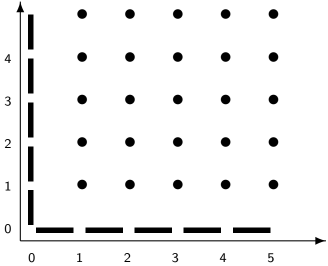

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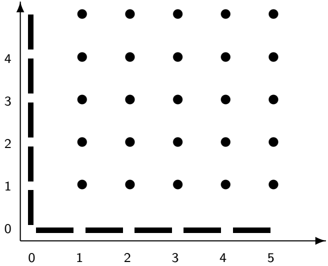

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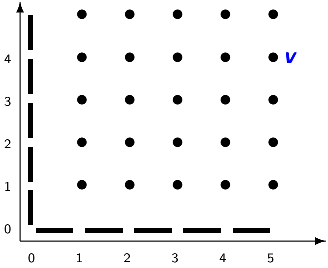

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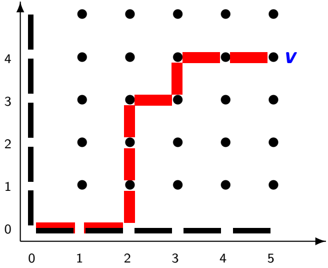

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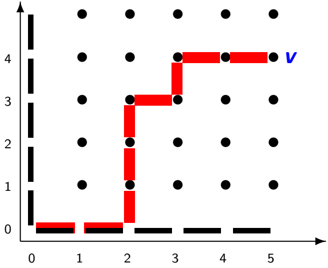

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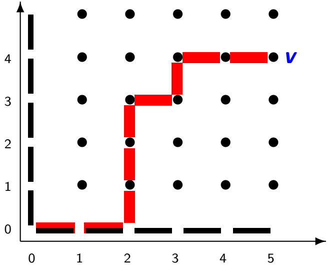
Define partition functions

Stationary log-gamma polymer



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$$Z^{\alpha}_{\mathbf{0},v} = \sum_{x(\cdot):\,\mathbf{0}\to v} \left(\prod_{k=1}^{T} \tau^{-1}_{x(k-1),\,x(k)} \right) \cdot \left(\prod_{k=T+1}^{|v|_1} w^{-1}_{x(k)} \right)$$

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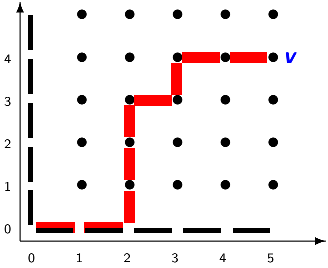

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where T is the exit time of the path from the boundary.

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Then $\{\log \tau_{x,y} : x, y \in \mathbb{Z}^2_+\}$ is a stationary cocycle with

$$\tau_{x,x+e_1} \sim \text{Gamma}(\boldsymbol{\alpha}) \quad \text{and} \quad \tau_{x,x+e_2} \sim \text{Gamma}(\rho - \boldsymbol{\alpha}).$$

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Tractable steady states useful for calculations and proofs.

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Recently Damron-Hanson used a weak limit of Busemann functions to get information about geodesics in FPP under weaker assumptions than those of Newman.

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$\{B^{\xi}(x, y)\}$ is a stationary cocycle with distributions

$$e^{-B^{\xi}(x,x+e_1)} \sim \text{Gamma}(\boldsymbol{\alpha}) \quad \text{and} \quad e^{-B^{\xi}(x,x+e_2)} \sim \text{Gamma}(\rho - \boldsymbol{\alpha})$$

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where direction ξ picks a parameter $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\xi) \in (0,\rho)$ via

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Proof. Couple with stationary version that gives control of ratios of $Z_{x,v}$.

Busemann functions solve variational formulas for log-gamma polymer

Recall $$g_{\text{pl}}(h) = \inf_{F} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \log \sum_{i=1,2} \frac{1}{2} e^{\omega_0 + h\cdot e_i + F(\omega,0,e_i)}$$

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Explicitly: $\quad g_{\text{pl}}(h) \;=\; h_1 - \Psi_0(\boldsymbol{\alpha}(\xi)) \;=\; h_2 - \Psi_0(\rho - \boldsymbol{\alpha}(\xi)).$

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Ingredients of stationary polymer defined from Busemann functions:

- boundary weights

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Busemann limit gives convergence of $\left\{\log Z_{0,\,x+v} - \log Z_{0,\,y+v}\right\}_{x,\,y}$ to a stationary cocycle, as $\mathbb{N}^2 \ni v \to \infty$.

Parameter $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\xi) \in (0,\rho)$ of the limiting stationary polymer process is selected by the spatial direction ξ of the limit.

Kardar-Parisi-Zhang (KPZ) fluctuations

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Limit distributions are found by converting probabilities into Fredholm determinants and performing asymptotic analysis. Combinatorics enters in the step to Fredholm determinants.

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Given parameter $\boldsymbol{\alpha}$ of the boundary, choose direction ξ from

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IID weights case:

Theorem For $1 \le p < 3/2$ $\exists\, C = C(p) < \infty$ such that

$$C^{-1} n^{p/3} \le \mathbb{E}\big| \log Z_{0,\lfloor n\xi \rfloor} - n g_{\mathrm{pp}}(\xi) \big|^p \le C\, n^{p/3}.$$

Busemann functions give limiting quenched polymer measure

In the stationary, ergodic environment $\{w_x, B^\xi(x,y)\}_{x,y\in\mathbb{Z}^2}$ define transition probability

$$\pi^\xi(x, x+e_i) = \frac{e^{-B^\xi(x,x+e_i)}}{e^{-B^\xi(x,x+e_1)} + e^{-B^\xi(x,x+e_2)}} = \frac{e^{-B^\xi(x,x+e_i)}}{w_x} \qquad i \in \{1,2\}$$

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Quenched polymer measure for n-paths from 0 with external field h:

$$Q^h_{0,(n)}(x_\bullet) = \frac{1}{Z^h_{0,(n)}} \left\{ \prod_{k=0}^{n-1} w_{x_k}^{-1} \right\} e^{h\cdot x_n}$$

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Theorem. If ξ is dual to h, then almost surely as $n \to \infty$ $Q^h_{0,(n)} \to P^\xi$ = random walk in a random environment (RWRE) with transition π^ξ.

Limiting RWRE

Proof.

$$
\begin{aligned}
Q^h_{0,(n)}(x_0,\dots,x_m) &= \frac{Z^h_{0,(n-m)}\circ T_{x_m}}{Z^h_{0,(n)}}\left\{\prod_{k=0}^{m-1} w_{x_k}^{-1}\right\} e^{h\cdot x_m} \\
&= e^{h\cdot x_m}\prod_{k=0}^{m-1}\frac{Z^h_{0,(n-k-1)}\circ T_{x_{k+1}}}{Z^h_{0,(n-k)}\circ T_{x_k}}\, w_{x_k}^{-1} \\
&\underset{n\to\infty}{\longrightarrow} e^{h\cdot x_m}\prod_{k=0}^{m-1} e^{-B^\xi(x_k,x_{k+1})-h\cdot(x_{k+1}-x_k)}\, w_{x_k}^{-1} \\
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Large deviation argument proves convergence of ratios of p2ℓ $Z^h_{0,(n)}$ to Busemann function selected by duality.

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Proof.

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$$= \; e^{h\cdot x_m} \prod_{k=0}^{m-1} \frac{Z^h_{0,(n-k-1)} \circ T_{x_{k+1}}}{Z^h_{0,(n-k)} \circ T_{x_k}} w_{x_k}^{-1}$$

$$\underset{n\to\infty}{\longrightarrow} \; e^{h\cdot x_m} \prod_{k=0}^{m-1} e^{-B^\xi(x_k,x_{k+1}) - h\cdot(x_{k+1}-x_k)} w_{x_k}^{-1}$$

$$= \; \prod_{k=0}^{m-1} \pi^\xi(x_k, x_{k+1}).$$

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*** Possible End ***

Variational formulas in terms of measures

Recall the limits:

$$g_{\mathrm{pl}}^{\beta}(h) \;=\; \lim_{n\to\infty}\; n^{-1}\,\log \sum_{(x_k)_{k=0}^{n}} 2^{-n}\,\exp\Big\{ \sum_{k=0}^{n-1} \omega_{x_k} \;+\; h\cdot x_n \Big\}$$

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Pick random step e_i, jump from $\eta = (\omega, z)$ to $S_{e_i}\eta = (T_z\eta, e_i)$.

Entropy relative to the Markov kernel

For measures $\boldsymbol{\mu}$ and transition kernels $\boldsymbol{q}$ on $\Omega \times \{e_1, e_2\}$, familiar relative entropy

$$H(\boldsymbol{\mu} \times \boldsymbol{q} \,|\, \boldsymbol{\mu} \times p) \;=\; \int_{\Omega \times \{e_1, e_2\}} \sum_{i=1,2} \boldsymbol{q}(\eta, S_{e_i}\eta) \log \frac{\boldsymbol{q}(\eta, S_{e_i}\eta)}{p(\eta, S_{e_i}\eta)} \, \boldsymbol{\mu}(d\eta).$$

Entropy relative to the Markov kernel

For measures μ and transition kernels q on $\Omega \times \{e_1, e_2\}$, familiar relative entropy

$$H(\mu \times q \,|\, \mu \times p) \;=\; \int_{\Omega\times\{e_1,e_2\}} \sum_{i=1,2} q(\eta, S_{e_i}\eta) \log \frac{q(\eta, S_{e_i}\eta)}{p(\eta, S_{e_i}\eta)} \, \mu(d\eta).$$

Let $\mu_0 = \Omega$-marginal of μ. Define

$$H_{\mathbb{P}}(\mu) = \begin{cases} \inf \left\{ H(\mu \times q \,|\, \mu \times p) : \mu q = \mu \right\} & \text{if } \mu_0 \ll \mathbb{P} \\ \infty & \text{otherwise.} \end{cases}$$

Infimum over Markov kernels q that fix μ.

Variational formulas

$\mathcal{M}_s(\Omega \times \{e_1, e_2\}) \;=\;$ space of invariant measures $\boldsymbol{\mu}$:

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Condition $\boldsymbol{\mu}_0 \ll \mathbb{P}$ removes compactness.