Small gaps between primes

James Maynard

CRM, Université de Montréal

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What is $\liminf_n (p_{n+1} - p_n)$*? In particular, is it finite?*

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We say a set $\mathcal{H}$ is *admissible* if for every prime p there is an integer n_p such that $n_p \not\equiv h \pmod{p}$ for all $h \in \mathcal{H}$.

Conjecture (Prime k-tuples conjecture)

Let $\mathcal{H} = \{h_1, \dots, h_k\}$ *be admissible. Then there are infinitely many integers* n*, such that all of* $n + h_1, \dots, n + h_k$ *are primes.*

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Corollary

Assume the prime k*-tuples conjecture. Then*

$$\liminf_n (p_{n+1} - p_n) = 2,$$
$$\liminf_n (p_{n+m} - p_n) \leq (1 + o(1)) m \log m.$$

Introduction II

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Theorem (Goldston, Pintz, Yıldırım, 2005)

$$\liminf_n \frac{p_{n+1}-p_n}{\log p_n} = 0.$$

This has recently been spectacularly extended by Zhang.

Theorem (Zhang, 2013)

$$\liminf_n (p_{n+1}-p_n) \le 70\,000\,000.$$

Small gaps between primes

Theorem (M. 2013)

The prime k-tuples conjecture holds for a positive proportion of admissible sets $\mathcal{H}$ of size k.

In particular:

1. $\liminf_n (p_{n+m} - p_n) \leq m^3 e^{4m+5}$ *for all* $m \in \mathbb{N}$.
2. $\liminf_n (p_{n+1} - p_n) \leq 600$.

Part (1) has also been independently proven by Terence Tao.
Our proof is independent of the methods of Zhang.

Overview

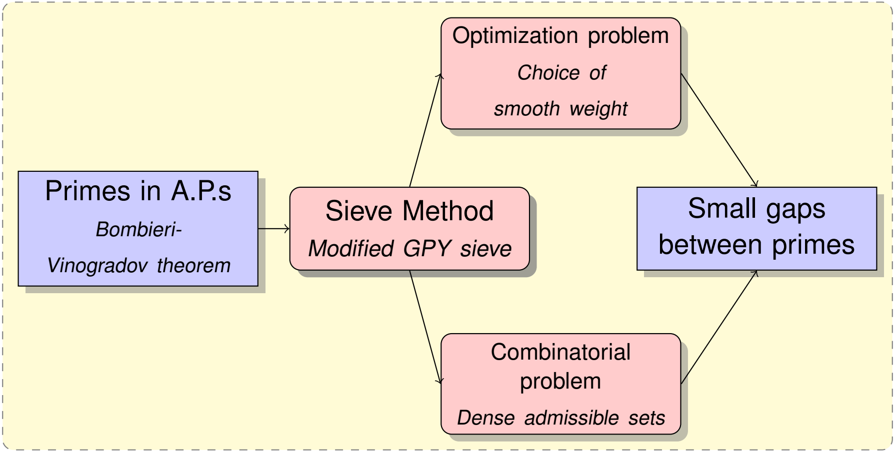

Figure : Outline of steps to prove small gaps between primes

Primes in arithmetic progressions

We use equidistribution results for primes in arithmetic progressions.

Heuristic

We believe that if $(a, q) = 1$ *then*

$$\pi(x; q, a) = \#\{p \leq x : p \equiv a \pmod q\} \approx \frac{\pi(x)}{\phi(q)}.$$

Let

$$E_q := \sup_{(a,q)=1} \Big| \pi(x; q, a) - \frac{\pi(x)}{\phi(q)} \Big|.$$

Primes in arithmetic progressions II

Definition

We say the primes have 'level of distribution θ' if, for any $A > 0$,

$$\sum_{q<x^\theta} E_q \ll_A \frac{x}{(\log x)^A}.$$

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Theorem (Bombieri-Vinogradov, 1965)

The primes have level of distribution θ for all $\theta < 1/2$.

Conjecture (Elliott-Halberstam, 1968)

The primes have level of distribution θ for all $\theta < 1$.

The GPY sieve

Given an admissible set $\mathcal{H} = \{h_1, \dots, h_k\}$, we estimate

$$S = \frac{\sum_{N \le n < 2N} \#\{1 \le i \le k : n + h_i \text{ prime}\} w_n}{\sum_{N \le n < 2N} w_n},$$

where w_n are non-negative weights (which we can choose freely).

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where w_n are non-negative weights (which we can choose freely). Then

1. If $S > m$, then at least one n makes a contribution $> m$.
2. Since $w_n \ge 0$, at least $m + 1$ of the $n + h_i$ are prime.
3. If $S > m$ for all large N, then $\liminf (p_{n+m} - p_n) < \infty$.

We need $S > 1$ for bounded gaps.

The GPY sieve II

Question

How do we choose w_n?

We choose w_n to mimic 'Selberg sieve' weights.

$$w_n = \Big(\sum_{d | \Pi(n), d < R} \lambda_d \Big)^2.$$

These depend on small divisors of $\Pi(n) = \prod_{i=1}^{k} (n + h_i)$.

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1. **Standard choice**: $\lambda_d = \mu(d)(\log R/d)^k$.
   We find $S \approx \theta$ if k large enough. Just fails to prove bounded gaps with $\theta = 1 - \epsilon$.

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2. **GPY choice**: $\lambda_d = \mu(d)f(d)$ for smooth f.
   We find $S \approx 2\theta$ if k is large enough. Just fails to prove bounded gaps with $\theta = 1/2 - \epsilon$.

Sieve weights

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- This is a discrete optimization problem - hard.
- Can test for optimality using Lagrangian multipliers.
- GPY choice **not** optimal - λ_d should be 'more arithmetic'.
- Arithmetic modifications of λ_d can do slightly better numerically, but difficult to analyze for general k.
- Although some heuristics behind GPY weights, the restrictions required by current methods are 'not natural'.

Key variation

New choice:

$$w_n = \Big(\sum_{\substack{d_1,\dots,d_k\\ d_i|n+h_i\\ \prod_{i=1}^k d_i<R}} \lambda_{d_1,\dots,d_k}\Big)^2, \qquad \lambda_{d_1,\dots,d_k} \approx \mu(\prod_{i=1}^k d_i) f(d_1,\dots,d_k).$$

We get extra flexibility in allowing our weights to depend on the divisors of each of the $n+h_i$ separately.

The $\lambda_{d_1,\dots,d_k}$ will be chosen in terms of a smooth function F, which we later optimize over.

For suitable F, can heuristically justify that these weights should be essentially optimal.

The sieve

We want to calculate sums

$$S_1 = \sum_{N<n\le 2N} w_n, \qquad S_{2,m} = \sum_{N<n\le 2N} \mathbf{1}_{n+h_m prime} w_n.$$

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Lemma

Let the primes have level of distribution $\theta > 0$. For suitable F

$$S_1 \sim c_{\mathcal{H}} N(\log N)^k I_k(F),$$

$$S_{2,m} \sim c_{\mathcal{H}} N(\log N)^k \frac{\theta}{2} J_{k,m}(F).$$

Technical simplification: restrict to $n \equiv v_p \pmod p$ for small primes. This means none of $n + h_i$ have small prime factors.

Selberg sieve calculations

Let's look at $S_{2,m}$.

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1. Open square and swap order of summation

$$S_{2,m} = \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k}} \lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k} \sum_{\substack{N<n\le 2N \\ d_i,e_i|n+h_i}} \mathbf{1}_{n+h_m prime}.$$

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2. The inner sum is a sum of primes in arithmetic progressions

$$\text{Inner sum} = \frac{\pi(2N)-\pi(N)}{\phi(q)} + O(E_q), \qquad q = \prod_{i=1}^{k}[d_i, e_i].$$

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   If $d_m = e_m = 1$ (and $(d_i, e_j) = 1$, also coprime to small primes)

3. Error terms are small using level-of-distribution results. $\lambda_{d_1,\dots,d_k}$ supported on $\prod_{i=1}^k d_i < N^{\theta/2}$ means $q < N^{\theta}$.

Selberg sieve calculations II

$$S_{2,m} \approx \frac{N}{\log N} \sum_{\substack{d_1,\dots,d_k \\ e_1,\dots,e_k \\ d_m=e_m=1}} \frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k [d_i, e_i]}$$

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4. Make a linear change of variables to diagonalize sum.

$$y^{(m)}_{r_1,\dots,r_k} \approx r_1 \dots r_k \sum_{r_i|d_i, d_m=1} \frac{\lambda_{d_1,\dots,d_k}}{d_1 \dots d_k}.$$

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$$S_{2,m} \approx \frac{N}{\log N} \sum_{\substack{r_1,\dots,r_k \\ r_m=1}} \frac{(y^{(m)}_{r_1,\dots,r_k})^2}{r_1 \dots r_k}.$$

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5. Similarly

$$S_1 \approx N \sum_{r_1,\dots,r_k} \frac{(y_{r_1,\dots,r_k})^2}{r_1\dots r_k}.$$

Selberg sieve calculations III

6. Relate $y^{(m)}$ variables to y variables

$$y^{(m)}_{r_1,\dots,r_k} \approx \sum_{a_m} \frac{y_{r_1,\dots,r_{m-1},a_m,r_{m+1},\dots,r_k}}{a_m}.$$

Selberg sieve calculations III

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7. Choose y variables to be a smooth function of $r_1,\dots,r_k$ and use partial summation.

$$y^{(m)} \approx \log R \int F(t_1,\dots,t_k)dt_m.$$

$$S_1 \approx N(\log R)^k I_k(F) = N(\log R)^k \int\dots\int F^2.$$

$$S_{2,m} \approx \frac{N(\log R)^{k+1}}{\log N} J_{k,m}(F) = \frac{N(\log R)^{k+1}}{\log N} \int\dots\int\Big(\int F dt_m\Big)^2.$$

Support conditions for λ met if $F(t_1,\dots,t_k)=0$ when $\sum_i t_i > 1$.

Reduce to smooth optimization

Choosing w_n in terms of a suitable function $F : \mathbb{R}^k \to \mathbb{R}$ gives

$$S = \frac{\theta J_k(F)}{2I_k(F)} + o(1).$$

Proposition

Let the primes have level of distribution θ and $\mathcal{H} = \{h_1, \dots, h_k\}$ be admissible. Let

$$M_k = \sup_F \frac{J_k(F)}{I_k(F)} = \frac{k \int \cdots \int \left(\int F(t_1, \dots, t_k) dt_1 \right)^2 dt_2 \dots dt_k}{\int \cdots \int F(t_1, \dots, t_k)^2 dt_1 \dots dt_k}.$$

If $M_k > 2m/\theta$ then there are infinitely many integers n such that at least $m+1$ of the $n + h_i$ are primes.

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If $M_k > 2m/\theta$ then there are infinitely many integers n such that at least $m+1$ of the $n + h_i$ are primes.

This has reduced our arithmetic problem (difficult) to a smooth optimization (easier).

Lower bounds for M_k

We want lower bounds for M_k.

1. To simplify, we let

$$F(t_1,\dots,t_k)=\begin{cases}\prod_{i=1}^k g(kt_i), & \text{if } \sum_{i=1}^k t_i<1,\\ 0, & \text{otherwise,}\end{cases}$$

for some function g.

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   for some function g.

2. If the center of mass of g^2 satisfies

$$\mu=\frac{\int_0^\infty tg(t)^2dt}{\int_0^\infty g(t)^2dt}<1$$

   then by concentration of measure we expect the restriction on support of F to be negligible.

Lower bounds for M_k II

3. If g is supported on $[0, T]$ we find that

$$M_k \geq \frac{(\int_0^T g(t)dt)^2}{\int_0^T g(t)^2 dt}\Big(1 - \frac{T}{k(1 - T/k - \mu)^2}\Big).$$

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4. For fixed μ and T, we can optimize over all such g by calculus of variations. We find the optimal g is given by

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5. With this choice of g, we find that a suitable choice of A, T gives

$$M_k > \log k - 2\log\log k - 2$$

if k is large enough.

Putting it all together

Proposition

1. $M_k > \log k - 2\log\log k - 2$ *if* k *is large enough.*
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Finally

Lemma

1. *There is an admissible set of size* k *contained in* $[0, H]$ *with* $H \approx k \log k$*.*
2. *We can take any* $\theta < 1/2$ *(Bombieri-Vinogradov).*

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2. *We can take any $\theta < 1/2$ (Bombieri-Vinogradov).*

These give

Theorem

$\liminf_n (p_{n+m} - p_n) \le Cm^3 e^{4m}$.

Hardy-Littlewood Conjecture

A simple counting argument shows a positive proportion of admissible sets satisfy the prime k-tuples conjecture for each k.

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1. If $k \gg_m 1$, then any admissible set $\mathcal{H}$ of size k contains a subset $\mathcal{H}' \subset \mathcal{H}$ of size m which satisfies prime m-tuples conjecture.
2. There are $\gg_k x^k$ admissible sets $\mathcal{H}$ of size k in $[0, x]^k$ (if $x \gg_k 1$).
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Hence

Theorem

There are $\gg_m x^m$ sets $\mathcal{H}' \subseteq [0, x]^m$ of size m satisfying the prime m-tuples conjecture if $x \gg_m 1$.

Other applications

Observation

*Since $M_k \to \infty$, we get bounded gaps for **any** $\theta > 0$.*

The method also works for any set of linear functions $a_i n + b_i$ instead of just shifts $n + h_i$. This makes the method very flexible.

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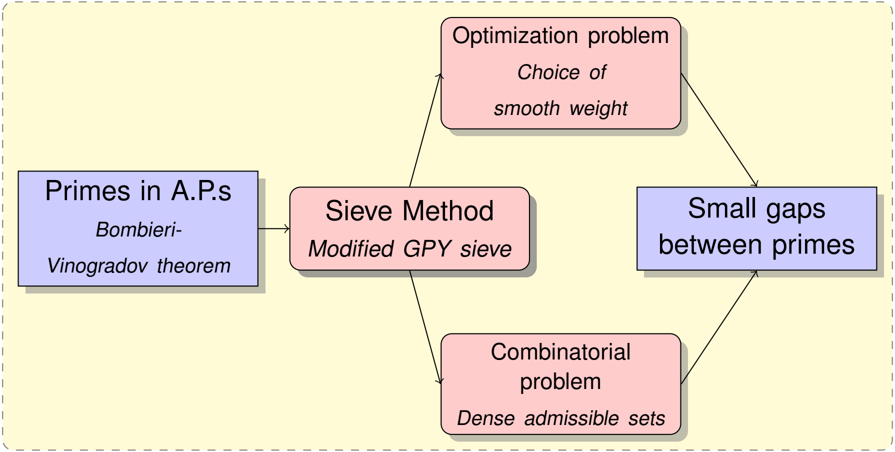
Strategy for proving close primes in subsets:

- Obtain an asymptotic in small residue classes (of Siegel-Walfisz type)
- Use a large sieve argument to show well distributed in residue classes $< x^\theta$.
- Use modified GPY sieve to show that there are primes close together.

How far can this go?

Polymath 8b is exploring how far these methods can go.



Improving primes in A.P.s

If we have better results about primes in arithmetic progressions, then we get stronger results.

Theorem (M. 2013)

Assume the primes have level of distribution θ for any $\theta < 1$. Then

$$\liminf_{n}(p_{n+1} - p_n) \leq 12.$$

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Assume the numbers with r prime factors have level of distribution θ for any $\theta < 1$ and any $r \in \mathbb{Z}$. Then

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Know barriers preventing this getting the twin prime conjecture. These weights 'fail by ϵ' analogously to Bombieri's sieve. Gaps of size 6 are the limit.

Modifications of the sieve

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Second result uses more modifications to the sieve to translate information more efficiently. This allows us to relax the restriction on the support of F.

- To estimate the terms weighted by $1_{n+h_m prime}$, we only required that $q = \prod_{i \neq m}[d_i, e_i] < N^{1-\epsilon}$.
- Under GEH, we can estimate S_1 using the above idea if $q = \prod_{i \neq m}[d_i, e_i] < N^{1-\epsilon}$ for **some** m.
- Even if we can't get an asymptotic for terms weighted by $1_{n+h_m prime}$, we can get a lower bound since

$$(\sum_{small} \lambda + \sum_{big} \lambda)^2 \geq \Big(\sum_{small} \lambda\Big)\Big(\sum_{small} \lambda + 2\sum_{big} \lambda\Big).$$

Improving primes in A.P.s II

Zhang/Polymath 8a have proven results about primes in APs which goes beyond $\theta = 1/2$.

- For large m, this gives an easy improvement

$$\liminf_n (p_{n+m} - p_n) \ll \exp((3.83)m).$$

- For small m, in principle this should give a numerical improvement, but this has not yet been incorporated into the current method in a strong enough form.

Sub-problems

Optimization problem:

- By pushing the small k computations further, we can show $\liminf_n (p_{n+1} - p_n) < 252$.
- Methods essentially optimal for large k. $M_k = \log k + O(1)$.

Sub-problems

Optimization problem:

- By pushing the small k computations further, we can show $\liminf_n(p_{n+1} - p_n) < 252$.
- Methods essentially optimal for large k. $M_k = \log k + O(1)$.

Combinatorial problem:

- Known optimal values for small k.
- Solution believed to be essentially optimal for large k.

..Or improve the sieve?

Questions

Thank you for listening.