Random constraint satisfaction problems: a point of view from physics

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Outline

1. Constraint satisfaction problems
2. Random ensembles of CSP
3. Statistical physics approach and results

Constraint satisfaction problems : definitions

n variables $\underline{x} = (x_1, \ldots, x_n) \in \mathcal{X}^n$ discrete alphabet $\mathcal{X}$

m constraints $\psi_a(\{x_i\}_{i \in \partial a}) = \begin{cases} 1 & \text{satisfied} \\ 0 & \text{unsatisfied} \end{cases}$

solutions $\mathcal{S} = \{\underline{x} : \psi_a(\underline{x}_{\partial a}) = 1 \ \forall a\}$

Constraint satisfaction problems : examples

- $\mathcal{X} = \{1, \dots, q\}$, $\psi_a(x_i, x_j) = \mathbb{1}(x_i \neq x_j)$
  on the edges $a = \langle i, j \rangle$ of a graph

  q-coloring problem

- $\mathcal{X} = \{\text{True}, \text{False}\}$, ψ_a depends on k variables $x_{i_a^1}, \dots, x_{i_a^k}$
  - $\psi_a = \mathbb{1}(z_{i_a^1} \vee \cdots \vee z_{i_a^1} = \text{True})$, with $z_i \in \{x_i, \overline{x}_i\}$

    k-satisfiability problem

  - $\psi_a = \mathbb{1}(x_{i_a^1} \oplus \cdots \oplus x_{i_a^k} = y_a)$, with $y_a \in \{\text{True}, \text{False}\}$

    k-xor-satisfiability problem

Constraint satisfaction problems

For a given instance (formula/graph), several questions :

- decision problem, is $|\mathcal{S}| > 0$? (said satisfiable if yes)
- counting problem, what is $|\mathcal{S}|$?
- optimization problem, what is $\max_{\underline{x}} \left[\sum_a \psi_a(\underline{x}) \right]$?

Worst-case complexity of the decision problem:

- k-xor-satisfiability easy (P) for all k
- k-satisfiability, q-coloring difficult (NP-complete) for $k, q \geq 3$

Random constraint satisfaction problems

What about their "typical case" complexities ?

"typical"= with high probability in some random ensemble of instances

Examples :

- coloring Erdös-Rényi random graphs $G(n, m)$
  choose m edges uniformly at random in the $\binom{n}{2}$ possible ones
- random (xor)satisfiability ensembles
  choose m hyperedges (k-uplets of variables), among $\binom{n}{k}$

Most interesting regime : $n, m \to \infty$ with $\alpha = m/n$ fixed

Random constraint satisfaction problems

$P_{n,\alpha} = \mathbb{P}$[random problem with n variables and $m = \alpha n$ constraints is satisfiable]

Obvious observations :

- $P_{n,\alpha}$ decreases with α
- $\lim_{n \to \infty} P_{n,\alpha} = 0$ for α large enough

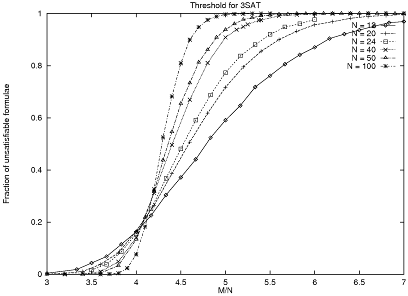
  First moment proof (for k-sat) :

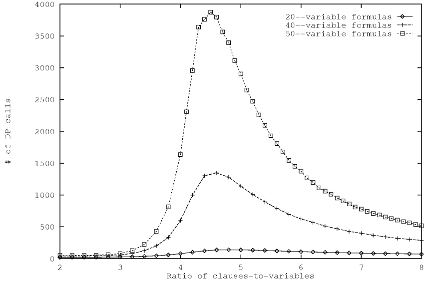
$$P_{n,\alpha} = \mathbb{P}[|\mathcal{S}| \geq 1] \leq \mathbb{E}[|\mathcal{S}|] = \sum_{\underline{x}} \mathbb{E}\left[\prod_{a=1}^{m} \psi_a(\underline{x})\right] = 2^n \left(1 - \frac{1}{2^k}\right)^{\alpha n} \to 0$$

  for $\alpha > \alpha_{\text{ub}}(k) = 2^k \ln 2$

Random constraint satisfaction problems

"Experimental" observation : phase transition for $1 - P_{n,\alpha}$





associated to a peak in the hardness of solving

Random constraint satisfaction problems

Phase transition conjecture : $\exists\ \alpha_{\rm s}(k)$ such that

$$\lim_{n\to\infty} P_{n,\alpha} = \begin{cases} 1 & \text{if } \alpha < \alpha_{\rm s}(k) \\ 0 & \text{if } \alpha > \alpha_{\rm s}(k) \end{cases}$$

Weaker version proven : $\exists\ \alpha_{\rm s}(k,n)$ such that [Friedgut]

$$\lim_{n\to\infty} P_{n,(1-\epsilon)\alpha_{\rm s}(k,n)} = 1 \ , \qquad \lim_{n\to\infty} P_{n,(1+\epsilon)\alpha_{\rm s}(k,n)} = 0$$

Early rigorous results for random k-satisfiability and q-coloring :

- upper and lower bounds on $\alpha_{\rm s}(k)$
  [Chao-Franco, Frieze-Suen, Achlioptas, Dubois, Boufkhad...]
- asymptotics of $\alpha_{\rm s}(k)$ at large k [Achlioptas, Moore, Naor, Peres]

$$\alpha_{\rm s}(k) \geq 2^k \ln 2 - k\frac{\ln 2}{2} + O(1)$$ [Achlioptas, Peres (04)]

Random constraint satisfaction problems

But :

- no precise value of $\alpha_s(k)$ for small k
- unsatisfactory understanding of algorithmic difficulty at $\alpha < \alpha_s(k)$

Why physics ?

Statistical mechanics :

- configuration space $\underline{x} = (x_1, \dots, x_n)$
- energy function $E(\underline{x})$
- temperature T
- Gibbs-Boltzmann distribution $\mu(\underline{x}) = \exp[-E(\underline{x})/T]/Z$

Low-temperature statistical physics $\approx$ combinatorial optimization

randomness in the distribution of instances $\approx$ disordered systems (spin glasses)

random graphs $\rightarrow$ mean-field diluted spin glasses

graph coloring corresponds to antiferromagnetic Potts model

Outcomes of the physics approach

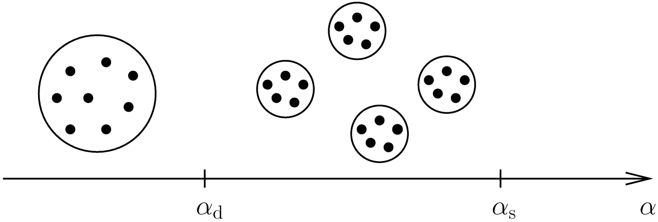
- quantitative estimation of $\alpha_{\rm s}(k)$
- refined picture of the satisfiable phase
- analysis of known algorithms
- suggestion of new ones

Refined picture of the satisfiable phase

Exponential number of solutions for $\alpha < \alpha_s$, $\sim$ exp[$ns(\alpha)$]

Clustering transition at another threshold $\alpha_d < \alpha_s$:

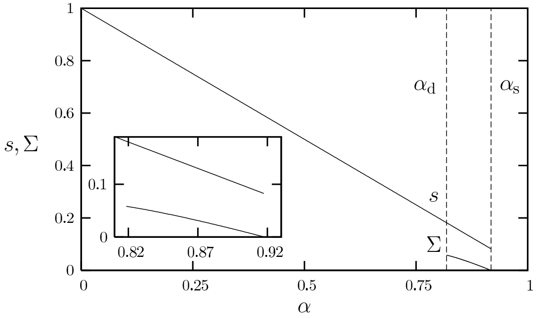
apparition of an exponential number of clusters ($\sim$ exp[$n\Sigma(\alpha)$]),
each containing an exponential number of solutions ($\sim$ exp[$ns_{\rm int}(\alpha)$])



at α_s cancellation of $\Sigma(\alpha)$, not of $s(\alpha)$

Can be proven rigorously for random xorsat [Creignou, Daudé] [Cocco, Dubois, Mandler, Monasson] [Mézard, Ricci-Tersenghi, Zecchina]

At α_{d}, apparition of a 2-core in the hypergraph



$$s(\alpha) = \Sigma(\alpha) + s_{\mathrm{int}}(\alpha)$$
$$\Sigma(\alpha_{\mathrm{s}}) = 0$$

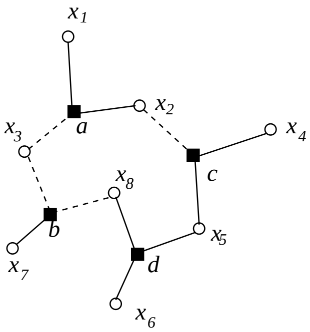
[more complicated picture for sat and col]

Methods

If the formula F has solutions,

define $\mu(\underline{x}) = \frac{1}{Z} \prod \psi_a(\underline{x}_{\partial a})$ uniform measure on $\mathcal{S}$

Factor graph representation of a formula :



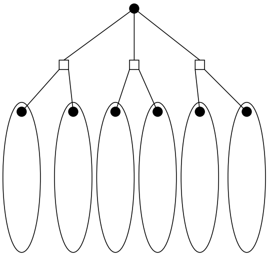
Crucial property : in the $n, m \to \infty$ limit with $\alpha = m/n$ fixed
local convergence of the factor graph to a random Galton-Watson tree

Methods

For a tree factor graph $\mu(\underline{x})$ is a rather simple object

(Belief Propagation is exact)

All marginal probabilities can be easily computed recursively :



Sparse random graphs converge locally to trees

Is it enough for their $\mu(\underline{x})$ to converge locally to the measure on the associated tree ?

It depends... (on the correlations decay)

Methods

- Yes in "replica symmetric" cases
  - Ferromagnetic Ising models [Dembo, Montanari]
  - Matchings [Bordenave, Lelarge, Salez]
  - Random CSP for $\alpha < \alpha_{\rm d}$

Removing a finite subtree, the variables at the boundary are asymptotically independent

Allows to compute in particular $\lim \frac{1}{n} \log Z$ entropy of solutions

Methods

- No in presence of “replica symmetry breaking” $(\alpha_{\rm d} < \alpha < \alpha_{\rm s})$

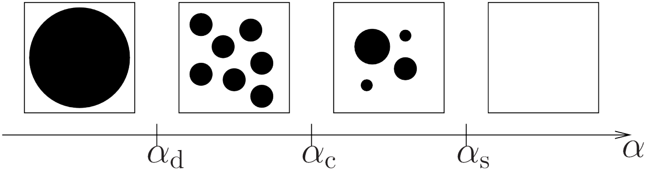
  Configuration space partitioned in clusters $\Rightarrow$ long-range correlations

  Removing a finite subtree, the variables at the boundary remains correlated, self-consistent ansatz on these boundary conditions

  With $\mu(\underline{x}) = \sum_c w_c \, \mu_c(\underline{x})$, each μ_c has short-range correlations, can be treated as above

  Properties of w_c encode the value of $\Sigma(\alpha)$, hence allows the computation of $\alpha_{\rm s}$

Clustering for k-sat and q-col



Further "condensation" transition :

exponential number of clusters only for $\alpha \in [\alpha_\mathrm{d}, \alpha_\mathrm{c}]$

In general $\alpha_\mathrm{c} < \alpha_\mathrm{s}$, for XORSAT they coincide

Recent rigorous results

Physical intuition and heuristic results turned into rigorous proofs

- Existence of clusters and frozen variables [Achlioptas, Molloy]
- Improved bounds on $\alpha_{\rm s}(k)$ at large k

$$\alpha_{\rm s}(k) \geq 2^k \ln 2 - \frac{3 \ln 2}{2} + o(1) \quad \text{[Coja-Oghlan, Panagiotou (12)]}$$

$$\alpha_{\rm s}(k) = 2^k \ln 2 - \frac{1 + \ln 2}{2} + o(1) \quad \text{[Coja-Oghlan (14)]}$$

- independence number of d-regular graphs for large (but finite) d [Ding, Sly, Sun (13)]

Perspectives

Schemes of rigorous proofs:

- Large k, q results, of combinatorics nature
  [Achlioptas, Coja-Oghlan]

- Interpolation method, originally devised for the Sherrington-Kirkpatrick model
  [Guerra, Toninelli, Aizenman, Talagrand, Panchenko]

- Local convergence approach [Aldous]
  - done for sparse random graphs in the “replica symmetric” regime
  - open problem in the “replica symmetry breaking” regime

Short bibliography

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