The many forms of rigidity for symplectic embeddings

based on work of Dan Cristofaro-Gardiner, Michael Entov, David Frenkel, Janko Latschev, Dusa McDuff, Dorothee Müller, FS, Misha Verbitsky

The problems

1. $\mathrm{E}(1, a) \stackrel{s}{\hookrightarrow} \mathrm{Z}^4(A)$
2. $\mathrm{E}(1, a) \stackrel{s}{\hookrightarrow} \mathrm{C}^4(A)$
3. $\mathrm{E}(1, a) \stackrel{s}{\hookrightarrow} \mathrm{P}(A, bA), \quad b \in \mathbb{N}$
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where:

$$\mathrm{E}(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} < 1 \right\}$$

can assume: $b = 1, \; a \geqslant 1$

and :

$$\begin{aligned} \mathrm{Z}^4(A) &= \mathrm{D}(A) \times \mathbb{C} \\ \mathrm{C}^4(A) &= \mathrm{P}(A, A) \\ \mathrm{P}(A, bA) &= \mathrm{D}(A) \times \mathrm{D}(bA) \\ \mathrm{T}^4(A) &= \mathrm{T}^2(A) \times \mathrm{T}^2(A) \end{aligned}$$

moment map images (under $(z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$) :

The answers

The answers

1. Gromov 1985: $\mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{Z}^4(A)$ iff $A \geqslant 1$

Hence

$$c_{\mathrm{EZ}}(a) \; := \; \inf \left\{ A \mid \mathrm{E}(1,a) \overset{s}{\hookrightarrow} \mathrm{Z}^4(A) \right\} \; \equiv 1$$

TOTAL symplectic rigidity, NO structure

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First: Truncate all down to $C^4(A)$:

$$\begin{aligned} c_1(a) &:= \inf\left\{A \mid E(1,a) \overset{s}{\hookrightarrow} C^4(A)\right\} \\ &\geqslant \sqrt{\frac{a}{2}} \quad \text{(Volume constraint)} \end{aligned}$$

2. Frenkel–Müller 2014, based on McDuff–S 2012:

$c_1(a)$ starts with the **Pell stairs** :

Pell numbers: $P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2}$

HC Pell numbers: $H_0 = 1,\ H_1 = 1,\ H_n = 2H_{n-1} + H_{n-2}$

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Form the sequence

$$
\begin{aligned}
(\gamma_1, \gamma_2, \gamma_3, \dots) \quad &:= \quad \left(\frac{P_1}{H_0},\ \frac{H_2}{2P_1},\ \frac{P_3}{H_2},\ \frac{H_4}{2P_3},\ \dots \right) \\
&= \quad \left(1, \quad \frac{3}{2}, \quad \frac{5}{3}, \quad \frac{17}{10}, \dots \right) \\
&\to \quad \frac{\sigma}{\sqrt{2}} \quad \text{where } \sigma = \sqrt{2} + 1 \ \text{ the silver ratio}
\end{aligned}
$$

An application:

Biran 1996:

k	1	2	3	4	5	6	7	$\geqslant 8$
p_k	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{48}{49}$	$\frac{224}{225}$	1

where $p_k =$ percentage of volume of $[0,1]^4 \subset \mathbb{R}^4$ that can be symplectically filled by k disjoint equal balls

What are these numbers?

The function c_1 explains Biran's list:

$$d_k := \inf \left\{ A \;\middle|\; \coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{C}^4(A) \right\}$$

Since $p_k = \dfrac{k \cdot \frac{1}{2}}{d_k^2}$, his list becomes

k	1	2	3	4	5	6	7	$\geqslant 8$
d_k	1	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{4}$	$\frac{15}{8}$	$\sqrt{\frac{k}{2}}$

and (McDuff 2009): $d_k = c_1(k)$, ie

$$\coprod_k \mathrm{B}^4(1) \overset{s}{\hookrightarrow} \mathrm{C}^4(A) \iff \mathrm{E}(1,k) \overset{s}{\hookrightarrow} \mathrm{C}^4(A)$$

Hence: It was worthwhile to elongate the domain:

$$B^4(1) \rightsquigarrow E(1, a)$$

Now: Also elongate the target:

$$C^4(A) = P(A, A) \rightsquigarrow P(A, bA), \quad b \geqslant 1$$

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For $a, b \geqslant 1$

$$c_b(a) \;=\; \inf \left\{ A \mid E(1, a) \overset{s}{\hookrightarrow} P(A, bA) \right\}$$

1-parametric family of problems

Volume constraint: $c_b(a) \geqslant \sqrt{\dfrac{a}{2b}}$

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Remarks

1. "The same" holds true for $b \in \mathbb{R}_{\geqslant 2}$ (no proof)

2. The first two linear steps, and the affine step of the Pell stairs are stable, the other steps disappear

Open problem: How does the Pell stairs disappear?
Understand $c_{1+\varepsilon}$

4. $E(1,a) \overset{s}{\hookrightarrow} T^4(A)$:

Note: $C^4(A)$ compactifies to both $S^2(A) \times S^2(A)$ and $T^4(A)$

Fact:
$$\begin{aligned} c_1(a) &= \inf \left\{ A \mid E(1,a) \overset{s}{\hookrightarrow} C^4(A) \right\} \\ &= \inf \left\{ A \mid E(1,a) \overset{s}{\hookrightarrow} S^2(A) \times S^2(A) \right\} \end{aligned}$$

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$$= \inf\left\{A \mid E(1,a) \overset{s}{\hookrightarrow} S^2(A) \times S^2(A)\right\}$$

On the other hand (Latschev–McDuff–S, Entov–Verbitsky 2014):

$E(1,a) \overset{s}{\hookrightarrow} T^4(A)$ whenever $\operatorname{Vol} E(1,a) < \operatorname{Vol} T^4(A)$

("Total flexibility")

But:

• It is unkown wether $\mathrm{Emb}(\mathrm{E}(1,a),\mathrm{T}^4(A))$ is connected (hence $<$)

• Hidden rigidity (Biran): Assume that

$$\varphi\colon\ \mathrm{B}^4(a) \overset{s}{\hookrightarrow} \mathrm{T}^4(1), \qquad \mathrm{Vol}\,\mathrm{B}^4(a) = \tfrac{3}{4},$$
$$\psi\colon\ \textstyle\coprod_2 \mathrm{B}^4(b) \overset{s}{\hookrightarrow} \mathrm{T}^4(A), \qquad \mathrm{Vol}\textstyle\coprod_2 \mathrm{B}^4(b) = \tfrac{2}{3}.$$

Then $\mathrm{Im}\,\psi \subset \mathrm{Im}\,\varphi$ is impossible (by Gromov's 2-ball theorem)

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Common ingredient:

from $\overline{\mathrm{B}}^4(a) \overset{s}{\hookrightarrow} (M^4, \omega)$ get symplectic form ω_a on the blow-up

$$\pi \colon M_1 \to M$$

in class
$$\pi^*[\omega] - a\, e$$

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Conversely: If $\pi^*[\omega] - a\, e$ has a symplectic representative,
non-degenerate along Σ,
then can "blow-down", get $B^4(a) \stackrel{s}{\hookrightarrow} (M, \omega_?)$

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omit ε, δ ...

compactify $C^4(1) = D(1) \times D(1)$ to $M = S^2(1) \times S^2(1)$

$\Delta := \{(z,z)\} \subset S^2 \times S^2$: holomorphic

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$\Delta := \{(z,z)\} \subset S^2 \times S^2$: holomorphic

blow-up M twice "at the right points" by size $\lambda = \frac{1}{3}$

get holomorphic sphere $\Delta_2 \subset M_2$ in class

$$[\Delta_2] = S_1 + S_2 - E_1 - E_2$$

and a chain of spheres $C_1 \cup C_2$ "bounding" $E(\lambda, 2\lambda)$

and a symplectic form ω_λ in class

$$[\omega_\lambda] = s_1 + s_2 - \lambda(e_1 + e_2)$$

Wish to "inflate" the form

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to

$$\omega_1 \ \text{ in } \ [\omega_1] = s_1 + s_2 - (e_1 + e_2)$$

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Inflation along $e_1 + e_2$ is impossible, but can inflate along

$$PD(\Delta_2) = s_1 + s_2 - e_1 - e_2 :$$

$\forall\, \tau > 0$ there exists a symplectic form Ω_τ in class

$$[\omega_\lambda] + \tau\, PD[\Delta_2] \;=\; (1+\tau)(s_1+s_2) - (\lambda+\tau)(e_1+e_2)$$

Get $\quad E(\lambda+\tau, 2(\lambda+\tau)) \overset{s}{\hookrightarrow} C^4(1+\tau)$

Hence $\quad E(1,2) \overset{s}{\hookrightarrow} C^4(\frac{1+\tau}{\lambda+\tau})$

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Cannot do inflation, since there are no J-curves to inflate along

But: Use existence of Kähler forms on blow-ups

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(M, J) Kähler

$H_J^{1,1}(M; \mathbb{R})$: classes represented by J-invariant closed 2-forms

candidates for Kähler classes: $\mathcal{C}_+^{1,1}(M, J) :=$

$$\left\{\alpha \in H_J^{1,1}(M; \mathbb{R}) \mid \alpha^m([V]) > 0 \ \forall \text{ complex subv. } V^m \subset (M, J)\right\}$$

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Hence: $\mathrm{E}(1,2) \overset{s}{\hookrightarrow} \mathrm{T}^4(1, 1+\varepsilon)$ (for ε irrational)
since there is a Kähler J without J-curves
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For $\mathrm{T}^4(1,1)$ use approximation of $(\mathrm{T}^4, \omega, I)$ by J close to I without J-curves
Kähler for some ω_J with $[\omega_J]$ close to $[\omega]$ (Kodaira–Spencer)

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"Yesterday": Entov–Verbitsky proved flexibility of ellipsoid packings of Kähler tori in all dimensions

by directly blowing-up $\mathrm{E}(a_1, \dots, a_n)$ (a_j relatively prime) and resolving the cyclic singularities (Hironaka)

Can assume $a \in \mathbb{Q}$

weight sequence

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Examples:

$$\mathbf{w}(3) = (1, 1, 1), \qquad \mathbf{w}\left(\frac{5}{3}\right) = \left(1, \frac{2}{3}, \left(\frac{1}{3}\right)^{\times 2}\right)$$

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McDuff, Frenkel–Müller:

$$E(1, a) \overset{s}{\hookrightarrow} P(\lambda, \lambda b) \iff$$

$$B(\mathbf{w}(a)) \coprod B(\lambda) \coprod B(\lambda b) \overset{s}{\hookrightarrow} B(\lambda(b+1))$$

Use three methods!

Method 1: obstructive classes

Method 2: dual version: reduction “at a point”

Method 3: ECH capacities

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(i) $\lambda \geqslant \sqrt{\frac{a}{2b}}$ volume constraint

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$$E(1, a) \stackrel{s}{\hookrightarrow} P(\lambda, \lambda b) \quad \Longleftrightarrow$$

(i) $\lambda \geqslant \sqrt{\frac{a}{2b}}$ volume constraint

(ii) $\lambda \geqslant \frac{\langle \mathbf{m}, \mathbf{w}(a) \rangle}{d + be}$ for all solutions $(d, e; \mathbf{m}) \in \mathbb{N}^3$ of

$$\sum_i m_i = 2(d+e) - 1, \qquad \sum_i m_i^2 = 2de + 1$$

constraint from J-spheres

For $b \geqslant 2$ all obstructions come from

$$E_n = (n, 1; 1^{\times(2n+1)}) \qquad \text{(linear steps)}$$

$$F_n = \big(n(n+1), n+1; n+1, n^{\times(2n+3)}\big) \qquad \text{(affine step)}$$

Method 3:

Hutchings, McDuff:

ECH capacities are complete invariants for our problem:

$$E(a,b) \overset{s}{\hookrightarrow} E(c,d) \iff c_k(E(a,b)) \leqslant c_k(E(c,d)) \quad \text{for all } k$$

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In particular

$$E(1,a) \overset{s}{\hookrightarrow} P(\lambda,\lambda b) \iff c_k(E(1,a)) \leqslant \lambda\, c_k(E(1,2b)) \quad \text{for all } k$$

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Cremona transform $\mathrm{Cr}\colon \mathbb{R}^{1+k} \to \mathbb{R}^{1+k}$

$$(\mu; \mathbf{a}) \mapsto (\mu+\delta;\ a_1+\delta,\ a_2+\delta,\ a_3+\delta,\ a_4,\ \ldots,\ a_k)$$

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Cremona move: reorder $\circ$ Cr

Tian-Jun Li, Buse–Pinsonnault:

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Dynamical interpretation:

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$\overline{\mathcal{P}^k_+}$ = positive cone:

$$\{(\mu;\mathbf{a}) \in \mathbb{R}^{1+k} \mid \mu, a_1, \ldots, a_k \geqslant 0,\ \|\mathbf{a}\| \leqslant \mu\}$$

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$\mathcal{R}$ = reduced vectors

Then $\mathcal{R} \subset \overline{\mathcal{C}(X_k)} \subset \overline{\mathcal{P}^k_+}$