Structure theorems for intertwining operators

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Wave operators

Let V real-valued potential in $\mathbb{R}^d$, bounded, sufficiently decaying, $H := -\Delta + V$, $H_0 := -\Delta$. Define

$$W_\pm := \lim_{t \to \mp\infty} e^{itH} e^{-itH_0}$$

Exists in the strong L^2-sense: $d \geq 3$, $f \in L^1 \cap L^2(\mathbb{R}^d)$, $V \in L^2$:

$$W_\pm f = f - i \int_0^\infty e^{itH} V e^{-itH_0} f \, dt$$

$$\int_1^\infty \left\| e^{itH} V e^{-itH_0} f \right\|_2 dt \leq \int_1^\infty \|V\|_2 \|e^{-itH_0} f\|_\infty \, dt$$

$$\lesssim \|V\|_2 \int_1^\infty t^{-\frac{d}{2}} \|f\|_1 \, dt < \infty$$

Unitarity of evolution, density of $L^1 \cap L^2(\mathbb{R}^d)$ in L^2 shows limit exists for all $f \in L^2$ and $W_\pm$ are isometries.

Intertwining property of wave operators

$f(H)W_\pm = W_\pm f(H_0)$, or

$$f(H)P = f(H)W_\pm W_\pm^* = W_\pm f(H_0) W_\pm^*,$$

with P orthogonal projection onto $\mathrm{Ran}(W_\pm)$. Easy to see: $\mathrm{Ran}(W_\pm) \perp L^2_{pp}$ (eigenfunctions of H).

Asymptotic completeness: $\mathrm{Ran}(W_\pm) = L^2_{ac}(\mathbb{R}^d)$, $L^2_{sc} = \{0\}$.

Agmon-Kato-Kuroda theory 1960s, early 70s: $|V(x)| \lesssim \langle x \rangle^{-1-\varepsilon}$ guarantees this, and no embedded eigenvalues in the continuous spectrum $[0, \infty)$. Short range condition.

Based on trace lemma: $\|\hat{f} \restriction S\|_{L^2(S)} \le C \|\langle x \rangle^\sigma f\|_{L^2(\mathbb{R}^d)}$, $\sigma > \frac{1}{2}$, where $S \subset \mathbb{R}^d$ compact hyper-surface (reduces to the case of a plane). Define restriction operator $\rho f := \hat{f} \restriction S$. Then $\rho^* g = \widehat{g\sigma_S}$, $\rho^* \rho\, f = \widehat{\sigma_S} * f$.

Weighted L^2 bound: $\|w \rho^* \rho\, w\, f\|_2 \le C(S) \|f\|_2$, $w(x) = \langle x \rangle^{-\frac{1}{2}-\varepsilon}$.

Limiting Absorption Principle

Same bound holds for the imaginary parts of the free resolvents

$$[(-\Delta-(\lambda^2+i0))^{-1}-(-\Delta-(\lambda^2-i0))^{-1}]f = c\lambda^{-1}\widehat{\sigma_{\lambda\mathbb{S}^{d-1}}} * f$$

For the full resolvent still true, Limiting Absorption Principle:

$$\|w(-\Delta-(\lambda^2+i0))^{-1}w f\|_2 \leq C(\lambda)\|f\|_2, \quad w(x)=\langle x\rangle^{-\frac{1}{2}-\varepsilon}$$

$C(\lambda)\to 0$ as $\lambda\to\infty$. Resolvent identity:

$$R(\lambda)=(H-(\lambda^2+i0))^{-1}=R_0(\lambda)+R_0(\lambda)VR(\lambda)=$$
$$=\ldots=R_0(\lambda)+R_0(\lambda)VR_0(\lambda)+R_0(\lambda)VR_0(\lambda)VR_0(\lambda)+\ldots$$

If V short range, small: $R(\lambda)$ inherits the limiting absorption principle. Split $V=|V|^{\frac{1}{2}}\,\mathrm{sign}(V)|V|^{\frac{1}{2}}=|V|^{\frac{1}{2}}U$.

Large V: $R(\lambda)=R_0(\lambda)+R_0(\lambda)|V|^{\frac{1}{2}}(I-UR_0(\lambda)|V|^{\frac{1}{2}})^{-1}UR_0(\lambda)$.

Resolvent and Fourier restriction

Inverse $(I - UR_0(\lambda)|V|^{\frac{1}{2}})^{-1} : L^2 \to L^2$ exists for $\lambda > 0$ (absence of embedded resonances and eigenvalues). For $\lambda = 0$ inverse might not exist: zero energy eigenvalue or resonance.

Stein-Tomas in place of trace lemma: If S has nonzero Gaussian curvature, then

$$\|\hat{f} \restriction S\|_{L^2(S)} \le C\|f\|_{L^{p_d}(\mathbb{R}^d)}, \qquad p_d = (2d+2)/(d+3)$$

Kenig-Ruiz-Sogge 87 established corresponding bound for $R_0(\lambda)$:

$$\|(-\Delta - (\lambda^2 + i0))^{-1}\|_{L^{p_d}(\mathbb{R}^d) \to L^{p_d'}(\mathbb{R}^d)} \le C\,\lambda^{-\frac{2}{d+1}}$$

Agmon-Kato-Kuroda theory in this setting:

$$M_q(f)(x) := \Big[\int_{|y|\le 1/2} |f(x+y)|^q\, dy\Big]^{\frac{1}{q}}, \quad q = \max(\frac{d}{2}, 1+)$$

$$\|V\|_Y := \sum_{j=0}^{\infty} 2^j \|V\|_{L^\infty(D_j)} < \infty, \quad M_q V \in L^{\frac{d+1}{2}}(\mathbb{R}^d) \qquad (\star)$$

Agmon-Kato-Kuroda via restriction theory

Theorem (Ionescu-S., 2004): *V real-valued, $V = V_1 + V_2$ with constituents satisfying either of conditions in $(\star)$, $\sigma_{ac} = [0, \infty)$, no singular continuous spectrum, pure point spectrum lies in $(-\infty, 0]$, discrete in $(-\infty, 0)$, eigenfunctions decay rapidly, wave operators $W_{\pm}$ exist and complete. Suitable limiting absorption principle holds.*

Magnetic potentials also admissible for this theorem. Goldberg-S. (2003): Stein-Tomas type limiting absorption principle for $L^{\frac{d}{2}}$ potentials $d = 3$; Ionescu-Jerison (2001): absence of embedded eigenvalues for $L^{\frac{d}{2}}$ potentials; Koch-Tataru 2005: absence of embedded evals under $(\star)$.
Condition $M_{\frac{d}{2}} V \in L^{\frac{d+1}{2}}(\mathbb{R}^d)$ weaker than $V \in L^{\frac{d}{2}}(\mathbb{R}^d)$ and sharp for $d \geq 3$. Ionescu-Jerison example $V \in L^p(\mathbb{R}^d), p > \frac{d+1}{2}$ with embedded evals, anisotropic decay $|V(x)| \simeq (1 + |x_1| + |x'|^2)^{-1}$

Intertwining operators for random potentials

Let $H_\omega = -\Delta_{\mathbb{Z}^d} + V_\omega$, $V_\omega(n) = \omega_n \langle n \rangle^{-\alpha}$, $\omega_n = \pm 1$ iid random.

Theorem (Bourgain, 2001): *$d = 2$, $\alpha > \frac{1}{2}$, $\tau > 0$, and $I \subset [-4+\tau, -\tau] \cup [\tau, 4-\tau]$. Then a.s. wave operators, restricted spectrally to I, i.e., $W_\pm(H_\omega, H_0)E_0(I)$, $W_\pm(H_0, H_\omega)E_\omega(I)$ exist and are complete.*

Relies on resolvent expansion, estimate (S.-Shubin-Wolff 2000)

$$\|\rho_2 V_\omega \chi_{[|n| \simeq N]} \rho_1^* \|_{L^2(S_1) \to L^2(S_2)} \lesssim N^{\frac{1}{2} - \alpha +} \quad (\dagger)$$

which high probability, S_1, S_2 curves in the plane. Bourgain's proof of $(\dagger)$ applies in any dimension, does not require curvature of S_1, S_2, uses dual Sudakov entropy bound in Banach spaces.

- analogy with trace lemma
- randomness reduces decay in deterministic theory by $\frac{1}{2}$ power.

Stein-Tomas in $\mathbb{R}^2$ combined with Bourgain's method yields same result for $V_\omega(n) = \omega_n\, v_n,\ v \in w_\varepsilon\, \ell^3(\mathbb{Z}^2),\ w_\varepsilon(n) = \langle n\rangle^{-\varepsilon}$.
Gap between point wise decay of $\langle n\rangle^{-\frac{1}{2}}$ and $\ell^3(\mathbb{Z}^2)$. Intrinsic problem with the key bound

$$\|\rho_2 V_\omega \chi_{[|n|\simeq N]} \rho_1^*\|_{L^2(S_1)\to L^2(S_2)} \lesssim 1$$

with high probability. TT^* on the left-hand side yields

$$\begin{aligned}
&\mathbb{E}\|\rho_2 V_\omega \chi_{[|n|\simeq N]} \rho_1^*\|^2_{L^2(S_1)\to L^2(S_2)} \\
&= \mathbb{E}\|\rho_2 V_\omega \chi_{[|n|\simeq N]} \rho_1^* \rho_1 V_\omega \chi_{[|n|\simeq N]} \rho_2^*\|_{L^2(S_2)\to L^2(S_2)} \\
&\geq \max_{\|f\|_{L^2(S_2)}=1} \mathbb{E}\langle V_\omega \chi_{[|n|\simeq N]} \rho_1^* \rho_1 V_\omega \chi_{[|n|\simeq N]} \rho_2^* f, \rho_2^* f\rangle \\
&= \max_{\|f\|_{L^2(S_2)}=1} \sum_{[|n|\simeq N]} \widehat{\sigma_{S_1}}(0)\, v_n^2 |F(n)|^2
\end{aligned}$$

Renormalizing random decaying potentials

Let f be Knapp example. Then $v \in \ell^3$ supported on a $N \times \sqrt{N}$ rectangle saturates the right-hand side.

"Renormalize" away self-energy interactions: $V_\omega \rightsquigarrow V_\omega + W$ with non-random "correction" $W_n = v_n^2 R_0(E + i0)$.

Likely that Born-expansion can be controlled at energy E via sharp restriction in the plane (Carleson-Sjölin/Zygmund L^4 bound).

- Consequences for a.c. spectrum of H_ω without correction W.
- Remove W after the fact? Problem here W just a little better than $L^2(\mathbb{Z}^2)$.

Yajima's L^p theory for the intertwining operator

In the 1990s Kenji Yajima showed that $W_\pm : L^p(\mathbb{R}^d) \to \mathbb{R}^d(\mathbb{R}^d)$, $1 \le p \le \infty$, $d \ge 3$, and $1 < p < \infty$, $d = 1, 2$. He needed to assume enough decay (and regularity in $d \ge 4$), and no zero energy eigenvalue/resonance. In dim=3 he needed $|V(x)| \le \langle x \rangle^{-5-\varepsilon}$. If zero energy singular, then $3/2 < p < 3$, $|V(x)| \le \langle x \rangle^{-6-\varepsilon}$.

Corollary: dispersive estimates for $e^{it\omega(H)} P_c(H)$ from those for $e^{it\omega(H_0)}$ via

$$e^{it\omega(H)} P_c(H) = W e^{it\omega(H_0)} W^*$$

Importance of 0 energy condition implied by this, too. For example, in dim=3

$$\left\| e^{itH} f \right\|_\infty \le \|W\|^2_{\infty\to\infty} \, C t^{-\frac{3}{2}} \|f\|_1, \qquad f \perp \text{bound states}$$

Possible issues: (i) strong assumptions on potential (ii) in some nonlinear applications 0 energy singularities do arise.

Yajima's proof, expansion of the wave operators

Iterate Duhamel with $f \in L^2$:

$$
\begin{aligned}
Wf &= f + W_1 f + \ldots + W_n f + \ldots, \\
W_1 f &= i \int_{t>0} e^{-it\Delta} V e^{it\Delta} f \, dt, \ \ldots \\
W_n f &= i^n \int_{t>s_1>\ldots>s_{n-1}>0} e^{-i(t-s_1)\Delta} V e^{-i(s_1-s_2)\Delta} V \ldots \\
&\quad e^{-is_{n-1}\Delta} V e^{it\Delta} f \, dt \, ds_1 \ldots ds_{n-1}
\end{aligned}
$$

Keel-Tao Strichartz endpoint (in $\mathbb{R}^3$)

$$
\| e^{itH_0} f \|_{L^2_t L^{6,2}_x} \lesssim \| f \|_{L^2}
$$

$$
\Big\| \int_{\mathbb{R}} e^{-isH_0} F(s) \, ds \Big\|_{L^2_x} \lesssim \| F \|_{L^2_t L^{6/5,2}_x},
$$

$V : L^{6,2}_x(\mathbb{R}^3) \to L^{6/5,2}_x(\mathbb{R}^3)$, $V \in L^{3,\infty}(\mathbb{R}^3)$
Dyson series converges in L^2 if $\|V\|_{3/2} \ll 1$.

Representations of the summands W_n

V, f, g Schwartz functions, $\varepsilon > 0$:

$$\langle W_n^\varepsilon f, g\rangle =$$

$$\frac{(-1)^n}{(2\pi)^3}\int_{\mathbb{R}^{3(n+1)}} \frac{\prod_{\ell=1}^n \widehat{V}(\xi_\ell - \xi_{\ell-1})\, d\xi_1 \ldots\, d\xi_{n-1}}{\prod_{\ell=1}^n (|\eta+\xi_\ell|^2 - |\eta|^2 + i\varepsilon)} \widehat{f}(\eta)\overline{\widehat{g}}(\eta+\xi_n)\, d\eta\, d\xi_n$$

$$\langle W_{1+}^\varepsilon f, g\rangle = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^6} \frac{\widehat{V}(\xi)}{|\eta+\xi|^2 - |\eta|^2 + i\varepsilon} \widehat{f}(\eta)\overline{\widehat{g}}(\eta+\xi)\, d\eta\, d\xi$$

$$= \int_{\mathbb{R}^6} K_1^\varepsilon(x, x-y) f(y)\, dy\, \overline{g}(x)\, dx$$

$$K_1^\varepsilon(x,z) = c|z|^{-2}\int_0^\infty e^{-is\hat{z}\cdot(x-z/2)}\widehat{V}(-s\hat{z})e^{-\varepsilon\frac{|z|}{2s}}\, s\, ds, \quad \hat{z} = z/|z|$$

$$K_1(x,z) = c|z|^{-2}L(|z| - 2x\cdot\hat{z}, \hat{z}), \;\; L(r,\omega) = \int_0^\infty \widehat{V}(-s\hat{z})e^{i\frac{rs}{2}}\, s\, ds$$

The structure of W_1 in $\mathbb{R}^3$

$S_\omega x := x - 2(\omega \cdot x)\omega$ reflection about plane $\omega^\perp$.

$$\begin{aligned}(W_1 f)(x) &= \int_0^\infty \int_{\mathbb{S}^2} L(r - 2\omega \cdot x, \omega) f(x - r\omega)\, dr d\omega \\ &= \int_{\mathbb{S}^2} \int_{\mathbb{R}} \mathbb{1}_{[r > -2\omega \cdot x]} L(r, \omega) f(S_\omega x - r\omega)\, dr d\omega \\ &= \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g_1(x, dy, \omega) f(S_\omega x - y)\, d\omega\end{aligned}$$

Therefore, with $\mathcal{H}^1_{\ell_\omega}$ Hausdorff measure on line along ω

$$g_1(x, dy, \omega) := \mathbb{1}_{[(y+2x)\cdot\omega > 0]} L(y \cdot \omega, \omega)\, \mathcal{H}^1_{\ell_\omega}(dy)$$

$$\int_{\mathbb{S}^2} \|g_1(x, dy, \omega)\|_{\mathcal{M}_y L^\infty_x}\, d\omega \le \int_{\mathbb{S}^2} \int_{\mathbb{R}} |L(r, \omega)|\, dr d\omega =: \|L\|$$

$$\|W_1 f\|_p \le \|L\| \|f\|_p$$

Bounding L

Define

$$\|f\|_{B^\beta} := \|\mathbb{1}_{[|x|\le 1]} f\|_2 + \sum_{j=0}^{\infty} 2^{j\beta} \left\| \mathbb{1}_{[2^j \le |x| \le 2^{j+1}]} f \right\|_2 < \infty$$

Then $\dot{B}^{\frac{1}{2}} \hookrightarrow L^{\frac{3}{2},1}(\mathbb{R}^3)$, $\dot{B}^{1} \hookrightarrow L^{\frac{6}{5},1}(\mathbb{R}^3)$, and

$$\|L(r,\omega)\|_{L^2_{r,\omega}} \lesssim \|V\|_{L^2}$$

$$\|L(r,\omega)\|_{L^1_{r,\omega}} \lesssim \sum_{k\in\mathbb{Z}} 2^{k/2} \|\mathbb{1}_{[2^k,2^{k+1}]}(|r|) L(r,\omega)\|_{L^2_{r,\omega}} \lesssim \|V\|_{\dot{B}^{\frac{1}{2}}} \lesssim \|V\|_{B^{\frac{1}{2}}}$$

Yajima showed for small potentials that $\|V\|_{B^{1+\varepsilon}} \ll 1$ implies

$$\|W_n f\|_p \le C^n \|V\|^n_{B^{1+\varepsilon}} \|f\|_p$$

which can be summed. For large potentials he incurred significant losses by terminating the expansion through the last term which contains perturbed evolution.

Structure Theorem I

Theorem (Beceanu-S. 16)

$V \in B^{1+}$ real-valued, zero energy regular for $H = -\Delta + V$. There exists $g(x, dy, \omega) \in L^1_\omega \mathcal{M}_y L^\infty_x$ with

$$\int_{\mathbb{S}^2} \|g(x, dy, \omega)\|_{\mathcal{M}_y L^\infty_x} \, d\omega < \infty$$

$$(W_+ f)(x) = f(x) + \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g(x, dy, \omega) f(S_\omega x - y) \, d\omega.$$

X Banach space of measurable functions on $\mathbb{R}^3$, invariant under translations and reflections, Schwartz functions are dense (or dense in Y with $X = Y^*$). Assume $\|\mathbb{1}_H f\|_X \le A \|f\|_X$ for all half spaces $H \subset \mathbb{R}^3$ and $f \in X$ with some uniform constant A. Then

$$\|W_+ f\|_X \le A C(V) \|f\|_X \qquad \forall \, f \in X$$

where $C(V)$ is a constant depending on V alone.

Structure Theorem II

Theorem (Beceanu-S. 16)

$V \in B^{1+2\gamma}$, $0 < \gamma$, with 0 energy hypothesis. Then

$$\int_{\mathbb{S}^2} \|g(x, dy, \omega)\|_{\mathcal{M}_y L_x^\infty} \, d\omega \le C_0 (1 + \|V\|_{B^{1+2\gamma}})^{38+\frac{105}{\gamma}} (1 + M_0)^{4+\frac{3}{\gamma}}$$

$$\sup_{\eta \in \mathbb{R}^3} \sup_{\varepsilon > 0} \left\| \big(I + R_0(|\eta|^2 \pm i\varepsilon)V\big)^{-1} \right\|_{\infty \to \infty} =: M_0 < \infty$$

C_0 absolute constant.

- 0 energy regular means that $\left\| \big(I + (-\Delta)^{-1} V\big)^{-1} \right\|_{\infty \to \infty} < \infty$.
- would be desirable to bound M_0 through this and size of V is some sense. Control of M_0 is not effective. See Rodnianski-Tao 2015, effective limiting absorption principles.
- Fall short by $\frac{1}{2}$ of scaling invariant class $\dot{B}^{\frac{1}{2}}$. First theorem also works in B^1, but lose quantitative control there.

Wiener algebra and inversion

We cannot sum the Dyson series. Instead we use Beceanu's operator-valued Wiener formalism. Recall classical Wiener theorem:

Proposition

Let $f \in L^1(\mathbb{R}^d)$. There exists $g \in L^1(\mathbb{R}^d)$ with

$$(1 + \hat{f})(1 + \hat{g}) = 1 \quad \text{on} \quad \mathbb{R}^d \tag{1}$$

iff $1 + \hat{f} \neq 0$ everywhere. Equivalently, there exists $g \in L^1(\mathbb{R}^d)$ so that

$$(\delta_0 + f) * (\delta_0 + g) = \delta_0 \tag{2}$$

iff $1 + \hat{f} \neq 0$ everywhere on $\mathbb{R}^d$. The function g is unique.

Two critical features (compactness as in Arzela-Ascoli):

- uniform L^1-modulus of continuity under translation.
- vanishing at ∞ in L^1 sense.

An operator-valued version

X Banach space, $\mathcal{W}_X$ algebra of bounded linear maps $T : X \to L^1(\mathbb{R}; X)$ with convolution

$$S * T(\rho)f = \int_{\mathbb{R}} S(\rho - \sigma)T(\sigma)f \, d\sigma$$

Adjoin unit, denote larger algebra $\widetilde{\mathcal{W}}_X$. Fourier transform satisfies

$$\sup_\lambda \|\hat{T}(\lambda)\|_{\mathcal{B}(X)} \le \|T\|_{\mathcal{W}_X}$$

Theorem (Beceanu 2009, Beceanu-Goldberg 2010)

Suppose $T \in \mathcal{W}_X$ satisfies

1. $\lim_{\delta \to 0} \|T(\rho) - T(\rho - \delta)\|_{\mathcal{W}_X} = 0.$
2. $\lim_{R \to \infty} \|T\chi_{|\rho| \ge R}\|_{\mathcal{W}_X} = 0.$

If $I + \hat{T}(\lambda)$ invertible in $\mathcal{B}(X)$ for all λ, then $\mathbf{1} + T$ possesses an inverse in $\widetilde{\mathcal{W}}_X$ of the form $\mathbf{1} + S$.

Wiener algebra and resolvents

Set $R_0^-(\lambda^2)(x) = (4\pi|x|)^{-1}e^{-i\lambda|x|}$, $\widehat{T^-}(\lambda) = VR_0^-(\lambda^2)$. Then

$$T^-(\rho)f(x) = (4\pi\rho)^{-1}V(x)\int_{|x-y|=\rho} f(y)\,dy \tag{3}$$

and thus

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |T^-(\rho)f(x)|\,dx\,d\rho &\leq \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(x)|}{|x-y|}|f(y)|\,dy\,dx \\ &\leq \frac{1}{4\pi}\|V\|_{\mathcal{K}}\|f\|_1.\end{aligned}$$

where $\|V\|_{\mathcal{K}} = \||x|^{-1} * |V|\|_\infty$. Algebra is $\mathcal{W}_{L^1}$, pointwise invertibility condition on Fourier side:

$$(I + VR_0^-(\lambda^2))^{-1} \in \mathcal{B}(L^1)$$

Spectral theory/zero energy assumption. Beceanu-Goldberg thus prove dispersive estimates for Schrödinger in $\mathbb{R}^3$ for $\|V\|_{\mathcal{K}} < \infty$.

Algebra for intertwining operators

The formulas for W_n suggest using three-variable kernels. Set

$$Z := \{ T(x_0, x_1, y) \in \mathcal{S}'(\mathbb{R}^9) \mid \mathcal{F}_y T(x_0, x_1, \eta) \in L^\infty_\eta L^\infty_{x_1} L^1_{x_0} \}$$

$$\|T\|_Z := \sup_{\eta \in \mathbb{R}^3} \|\mathcal{F}_y T(x_0, x_1, \eta)\|_{L^\infty_{x_1} L^1_{x_0}}$$

Operation $\circledast$ on $T_1, T_2 \in Z$

$$(T_1 \circledast T_2)(x_0, x_2, y) = \mathcal{F}^{-1}_\eta \Big[\int_{\mathbb{R}^3} \mathcal{F}_y T_1(x_0, x_1, \eta) \mathcal{F}_y T_2(x_1, x_2, \eta) \, dx_1 \Big](y)$$

Seminormed space $V^{-1}B$ defined as

$$V^{-1}B = \{ f \text{ measurable} \mid V(x) f(x) \in B^\sigma \}$$

with the seminorm $\|f\|_{V^{-1}B} := \|Vf\|_{B^\sigma}$. Set $X_{x,y} := L^1_y V^{-1} B_x$. Then $L^1_y L^\infty_x$ dense in $X_{x,y}$.

$X_{x,y}$ and Y spaces

Let Y be the space (algebra under $\circledast$) of three-variable kernels

$$Y := \Big\{ T(x_0, x_1, y) \in Z \mid \forall f \in L^\infty$$
$$(fT)(x_1, y) := \int_{\mathbb{R}^3} f(x_0)\, T(x_0, x_1, y)\, dx_0 \in X_{x_1,y} \Big\},$$

with norm

$$\|T\|_Y := \|T\|_Z + \|T\|_{\mathcal{B}(V^{-1}B_{x_0}, X_{x_1,y})}$$

For $\mathfrak{X} \in L^1_y L^\infty_x$, define *contraction* of $T \in Y$ by $\mathfrak{X}$ to be

$$(\mathfrak{X}T)(x, y) := \int_{\mathbb{R}^6} \mathfrak{X}(x_0, y_0)\, T(x_0, x, y - y_0)\, dx_0\, dy_0.$$

Then $\mathfrak{X}T \in X_{x,y}$, $\|\mathfrak{X}T\|_X \leq \|T\|_Y \|\mathfrak{X}\|_X$. This turns Y into an algebra.

Reason behind these structures: define

$$\mathcal{F}_y T^{\varepsilon}_{1+}(x_0, x_1, \eta) = e^{-ix_1\eta} R_0(|\eta|^2 - i\varepsilon)(x_0, x_1) V(x_0)\, e^{ix_0\eta}$$

$$T^{\varepsilon}_{2+} = T^{\varepsilon}_{1+} \circledast T^{\varepsilon}_{1+}, \;\; T^{\varepsilon}_{3+} = T^{\varepsilon}_{2+} \circledast T^{\varepsilon}_{1+} \;\; \text{etc.}$$

Then

$$\begin{aligned}\langle W^{\varepsilon}_{n+} f, g\rangle &= \frac{(-1)^n}{(2\pi)^3} \int_{\mathbb{R}^6} \mathcal{F}^{-1}_{x_0} \mathcal{F}_{x_n, y} T^{\varepsilon}_{n+}(0, \xi_n, \eta) \widehat{f}(\eta) \overline{\widehat{g}}(\eta + \xi_n)\, d\eta\, d\xi_n \\ &= (-1)^n \int_{\mathbb{R}^9} \mathcal{F}^{-1}_{x_0} T^{\varepsilon}_{n+}(0, x, y) f(x - y) \overline{g}(x)\, dy\, dx.\end{aligned}$$

as well as

$$\begin{aligned}&\langle W^{\varepsilon}_{+} f, g\rangle \\ &= \langle f, g\rangle - \frac{1}{(2\pi)^3} \int_{\mathbb{R}^6} \mathcal{F}^{-1}_{x_0} \mathcal{F}_{x_1, y} T^{\varepsilon}_{+}(0, \xi_1, \eta) \widehat{f}(\eta) \overline{\widehat{g}}(\eta + \xi_1)\, d\eta\, d\xi_n \\ &= \langle f, g\rangle - \int_{\mathbb{R}^9} \mathcal{F}^{-1}_{x_0} T^{\varepsilon}_{+}(0, x, y) f(x - y) \overline{g}(x)\, dy\, dx.\end{aligned}$$

Key invertibility problem

Here

$$\mathcal{F}_y T_{\pm}^{\varepsilon}(x_0, x_1, \eta) := e^{ix_0\eta}\big(R_V(|\eta|^2 \mp i\varepsilon)V\big)(x_0, x_1)e^{-ix_1\eta}$$

$T_{1+}^{\varepsilon}, T_{+}^{\varepsilon} \in Z$ and resolvent identity reads as follows:

$$(I + T_{1+}^{\varepsilon}) \circledast (I - T_{+}^{\varepsilon}) = (I - T_{+}^{\varepsilon}) \circledast (I + T_{1+}^{\varepsilon}) = I$$

We need to invert this in the smaller algebra Y, otherwise too little control of wave operators.

If $I + T_{1+}^{\varepsilon}$ is invertible in Y, hence in Z, its inverse is $I - T_{+}^{\varepsilon}$ both in Z and in Y, hence we obtain that $T_{+}^{\varepsilon} \in Y$ uniformly in $\varepsilon > 0$.

Small potentials in B^{1+}

Define Y with $\sigma \geq \frac{1}{2}$ fixed. Then

$$\sup_{\varepsilon>0} \| T^{\varepsilon}_{1+} \|_Y \lesssim \|V\|_{B^{\frac{1}{2}+\sigma}} \quad \text{whence by induction}$$

$$\sup_{\varepsilon>0} \| T^{\varepsilon}_{n+} \|_Y \leq C^n \|V\|^n_{B^{\frac{1}{2}+\sigma}} \quad \text{for all} \quad n \geq 1$$

and

$$(W_{n+} f)(x) = \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g^{\varepsilon}_n(x, dy, \omega) f(S_\omega x - y)\, d\omega$$

where for fixed $x \in \mathbb{R}^3$, $\omega \in \mathbb{S}^2$ the expression $g^{\varepsilon}_n(x, \cdot, \omega)$ is a measure satisfying

$$\sup_{\varepsilon>0} \int_{\mathbb{S}^2} \|g^{\varepsilon}_n(x, dy, \omega)\|_{\mathcal{M}_y L^\infty_x}\, d\omega \leq C^n \|V\|^n_{B^{\frac{1}{2}+\sigma}}$$

Recursive definition of the structure functions

Identifying operator W^ε_{n+} with its kernel one has

$$\begin{aligned} W^\varepsilon_{n+} &= (-1)^n \mathbb{1}_{\mathbb{R}^3} T^\varepsilon_{n+} = (-1)^n \mathbb{1}_{\mathbb{R}^3} (T^\varepsilon_{(n-1)+} \circledast T^\varepsilon_{1+}) \\ &= -((-1)^{n-1} \mathbb{1}_{\mathbb{R}^3} T^\varepsilon_{(n-1)+}) T^\varepsilon_{1+} = -W^\varepsilon_{(n-1)+} T^\varepsilon_{1+} \end{aligned}$$

Second line: contraction of a kernel in Y by an element of X.
Thus

$$\sup_{\varepsilon>0} \| W^\varepsilon_{n+} \|_X \leq \| \mathbb{1}_{\mathbb{R}^3} \|_{V^{-1}B} \sup_{\varepsilon>0} \| T^\varepsilon_{n+} \|_Y \leq C^n \| V \|^{n+1}_{B^{\frac{1}{2}+\sigma}} \qquad (4)$$

and with $f^\varepsilon_{y'}(x') = W^\varepsilon_{(n-1)+}(x', y')$ we have

$$g^\varepsilon_n(x, dy, \omega) := \int_{\mathbb{R}^3} g^\varepsilon_{1, f^\varepsilon_{y'}}(x, d(y - S_\omega y'), \omega)\, dy'$$

and $g^\varepsilon_{1, f^\varepsilon_{y'}}$ is the structure function for the potential $f^\varepsilon_{y'} V$.

Wiener theorem in Y

Proposition

$V \in B^\sigma$ with $\frac{1}{2} \leq \sigma < 1$, define Y with this σ, V. Suppose $S \in Y$ satisfies, for some $N \geq 1$

$$\lim_{\varepsilon \to 0} \|\varepsilon^{-3}\chi(\cdot/\varepsilon) * S^N - S^N\|_Y = 0$$

$$\lim_{L \to \infty} \|(1 - \hat{\chi}(y/L))S(y)\|_Y = 0$$

Assume $I + \hat{S}(\eta)$ has inverse in $\mathcal{B}(L^\infty)$ of the form $(I + \hat{S}(\eta))^{-1} = I + U(\eta)$, with $U(\eta) \in \mathcal{F}Y$ for all $\eta \in \mathbb{R}^3$, and uniformly so, i.e.,

$$\sup_{\eta \in \mathbb{R}^3} \|U(\eta)\|_{\mathcal{F}Y} < \infty$$

Finally, suppose $\eta \mapsto \hat{S}(\eta)$ is uniformly continuous $\mathbb{R}^3 \to \mathcal{B}(L^\infty)$. Then $I + S$ is invertible in Y under $\circledast$.

A scaling invariant condition

Schwartz V, set $\|\!|V|\!\| := \|L_V\|_{L^1_{t,\omega}}$. Recall

$$L_V(t,\omega) = \int_0^\infty \widehat{V}(-\tau\omega) e^{\frac{i}{2}t\tau}\, \tau \, d\tau$$

For any Schwartz function v in $\mathbb{R}^3$

$$\|v\|_B := \sup_{\Pi} \int_{-\infty}^{\infty} \|\!|\delta_{\Pi(t)}\, v(x)|\!\| \, dt$$

where Π is a 2-dimensional plane through the origin, and $\Pi(t) = \Pi + t\vec{N}$, $\vec{N}$ being the unit norm to Π. Then

$$\|v\|_B \lesssim \sup_{\omega \in \mathbb{S}^2} \int_{-\infty}^{\infty} \sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}} \|\psi(2^{-k}x')v(x'+s\omega)\|_{\dot{H}^{\frac{1}{2}}(\omega^\perp)} \, ds$$

This is finite on Schwartz functions.

A scaling theorem for small potentials

Theorem (Beceanu-S. 17)

There exists $c_0 > 0$ *so that for any real-valued* V *with* $\|V\|_B + \|V\|_{\dot{B}^{\frac{1}{2}}} \leq c_0$, *there exists* $g(x, y, \omega) \in L^1_\omega \mathcal{M}_y L^\infty_x$ *with*

$$\int_{\mathbb{S}^2} \|g(x, dy, \omega)\|_{\mathcal{M}_y L^\infty_x} \, d\omega \lesssim c_0$$

such that for any $f \in L^2$ *one has the representation formula*

$$(W_+ f)(x) = f(x) + \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g(x, dy, \omega) f(S_\omega x - y) \, d\omega.$$

No theorem for large scaling invariant potentials yet. Requires redoing all the spectral theory in this new norm.