Extremes of non-intersecting Brownian motions: from Yang-Mills theory to directed polymers

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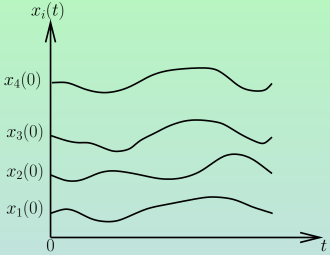
Non-intersecting Brownian motions

- N non intersecting Brownian motions in one-dimension

$$x_1(t) < x_2(t) < ... < x_N(t) \; , \quad \forall t \geq 0$$



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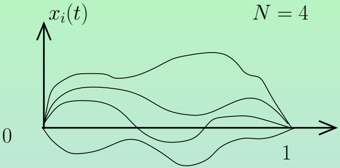

watermelons

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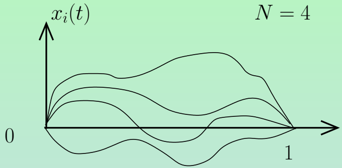
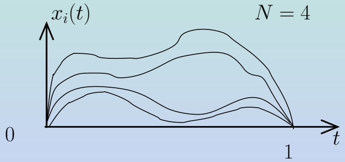


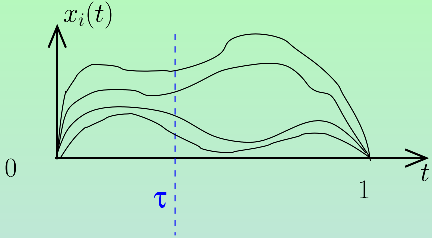
watermelons

Tracy & Widom '07



watermelons "with a wall"

Non-intersecting excursions and RMT



- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

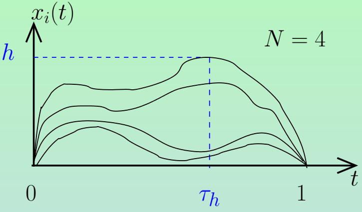
$$P_{\text{joint}}(\mathbf{x}, \tau) \propto \prod_{i=1}^{N} x_i^2 \prod_{1 \le i < j \le N} (x_i^2 - x_j^2)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^{N} x_i^2}$$

$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of the Bogoliubov-de Gennes type (class C)

A. Atland, M. R. Zirnbauer '96

Extremes of N non-intersecting excursions



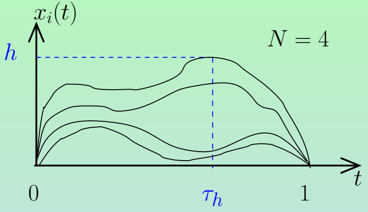
$$0 < x_1(t) < x_2(t) < ... < x_N(t)$$

$$\mathcal{H}_N = \max_{t\in[0,1]} x_N(t)$$

$$\mathcal{T}_{\mathcal{H}} = \arg\max_{t\in[0,1]} x_N(t)$$

1. Distribution of the maximum $F_N(h) = \mathbb{P}(\mathcal{H}_N \leq h)$
2. Joint probability density function of $\mathcal{H}_N, \mathcal{T}_{\mathcal{H}}$: $P_N(h, \tau_h)$

Extremes of N non-intersecting excursions



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In this talk :

- Exact computation for $F_N(h)$ and $P_N(h, \tau_h)$ for all N
- Large N asymptotics (typical and large fluctuations)

Outline

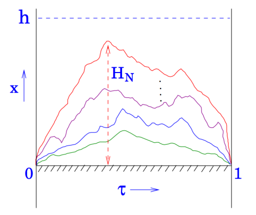
1. PDF of the maximum, Yang-Mills theory and 3$^{\rm rd}$ order transition
2. Applications to directed polymers in random media
3. Relation with determinantal formulas from Airy$_2$ process
4. Conclusion

Outline

An exact expression for $F_N(h)$

- Distribution of the maximal height $\mathcal{H}_N$

$$F_N(h) \quad = \quad \mathbb{P}\left[x_N(\tau) \leq h \,,\, \forall\, 0 \leq \tau \leq 1\right]$$



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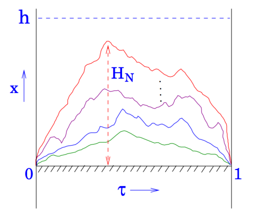


- Exact result for finite N G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling '08

$$F_N(h) = \frac{A_N}{h^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2}\sum_{i=1}^N n_i^2}$$

$$A_N = \frac{\pi^{2N^2+N}}{2^{N^2+\frac{N}{2}} \prod_{j=0}^{N-1} \Gamma(2+j)\Gamma(\frac{3}{2}+j)}$$ see also T. Feierl, M. Katori *et al.* '08

Connection to Yang-Mills theory in 2d

- Partition function of Yang-Mills theory on a 2d manifold $\mathcal{M}$ with a gauge group G, described by a gauge field $A_\mu(x) \equiv A^a_\mu(x) T^a$, $\mu \in \{1, 2\}$

$$\mathcal{Z}_{\mathcal{M}} = \int [\mathcal{D}A_\mu] e^{-\frac{1}{4\lambda^2} \int \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] \sqrt{g} d^2 x}$$

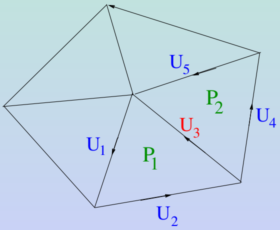
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

Ex: $G \equiv SU(2)$: electro-weak interact$^{\mathrm{o}}$, $G \equiv SU(3)$: chromodynamics

- Regularization on the lattice

$$\mathcal{Z}_{\mathcal{M}} = \int \prod_L dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$

$$U_P = \prod_{L \in \text{plaquette } P} U_L$$



Heat-kernel action

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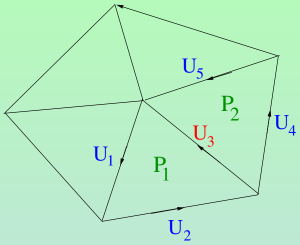

- A common choice : Wilson's action   Wilson'74, Gross & Witten '80, Wadia '80

$$Z_P(U_P) = \exp\left[b\, N\, \mathrm{Tr}(U_P + U_P^\dagger)\right]$$

Heat-kernel action

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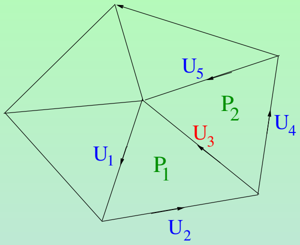

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$$Z_P(U_P) = \exp\left[b\, N\, \mathrm{Tr}(U_P + U_P^\dagger)\right]$$

- Alternative choice : invariance under decimation $\Rightarrow$ Migdal's recursion relation

$$\int dU_3\, Z_{P_1}(U_1 U_2 U_3) Z_{P_2}(U_4 U_5 U_3^\dagger) = Z_{P_1+P_2}(U_1 U_2 U_4 U_5)$$

$$Z_P(U_P) = \sum_R d_R \chi_R(U_P) \exp\left[-\frac{A_P}{2N} C_2(R)\right]$$

Migdal'75, Rusakov'90

Partition function of Yang-Mills theory on the 2d-sphere

- Partition funct° on $\mathcal{M}$, of genus g, computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^{2-2g} \exp\left[-\frac{A}{2N} C_2(R)\right]$$

Partition function of Yang-Mills theory on the $2d$-sphere

- Partition funct° on the sphere computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_{R} d_R^2 \exp\left[-\frac{A}{2N} C_2(R)\right]$$

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$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^2 \exp\left[-\frac{A}{2N} C_2(R)\right]$$

- Irreducible representations R of G are labelled by the lengths of the Young diagrams:
  - If $G = \mathrm{U}(N)$

$$\mathcal{Z}_{\mathcal{M}} = c_N\, e^{-A\frac{N^2-1}{24}} \sum_{n_1,\ldots,n_N=0}^{\infty} \prod_{i<j} (n_i - n_j)^2 e^{-\frac{A}{2N}\sum_{j=1}^N n_j^2}$$

  - If $G = \mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \hat{c}_N\, e^{A\,(N+\frac{1}{2})\,\frac{N+1}{12}} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{j=1}^{N} n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N}\sum_{j=1}^N n_j^2}$$

Correspondence between YM$_2$ on the sphere and watermelons

- Partition function of YM$_2$ on the sphere with gauge group $\mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \mathcal{Z}(A; \mathrm{Sp}(2N))$$

$$\mathcal{Z}(A; \mathrm{Sp}(2N)) = \hat{c}_N \, e^{A\,(N+\frac{1}{2})\,\frac{N+1}{12}} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{j=1}^{N} n_j^2 \right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N} \sum_{j=1}^{N} n_j^2}$$

- Cumulative distribution of the maximal height of watermelons with a wall

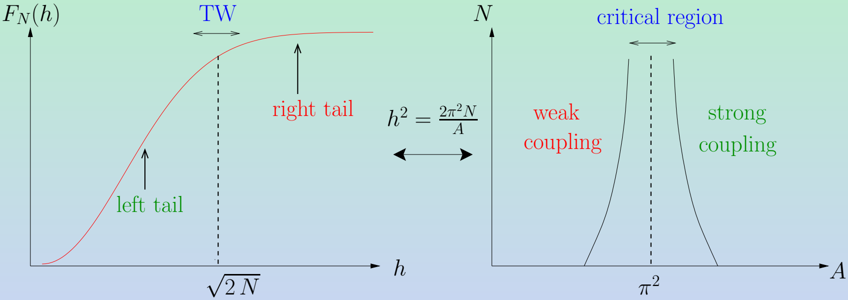
$$F_N(h) = \frac{A_N}{h^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \left(\prod_{j=1}^{N} n_j^2 \right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{\pi^2}{2h^2} \sum_{j=1}^{N} n_j^2}$$

$$\propto \mathcal{Z}\left(A = \frac{2\pi^2 N}{h^2}; \mathrm{Sp}(2N) \right)$$

P. J. Forrester, S. N. Majumdar, G. S. '11

Large N limit of YM_2 and consequences for $F_N(M)$

- Weak-strong coupling transition in YM_2 Douglas-Kazakov '93



Large N asymptotics: discrete Coulomb gas

- Saddle point analysis for large N, $h = \tilde{h}\sqrt{2N}$

$$F_N(h) \propto \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\le j<k\le N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{4\tilde{h}^2 N}\sum_{i=1}^N n_i^2}$$

when $N \to \infty$, $\dfrac{n_i}{2N} := x_i$ are continuous variables

$$F_N(\tilde{h}\sqrt{2N}) \sim \int \mathcal{D}\tilde{\rho}(x) e^{-N^2 S[\tilde{\rho}]} \; , \; \tilde{\rho}(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - x_i) \le 2$$

$$S[\tilde{\rho}] = \frac{\pi^2}{\tilde{h}^2}\int_0^a dx\, x^2 \tilde{\rho}(x) - \int_0^a dx \int_0^a dx' \,\tilde{\rho}(x)\tilde{\rho}(x') \ln|x^2 - x'^2|$$

Large N asymptotics: saddle point analysis

- Constrained saddle point

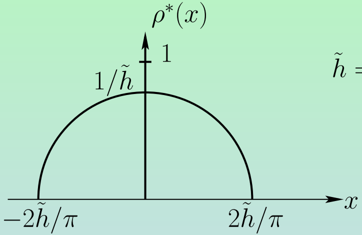
$$\int \mathcal{D}\tilde{\rho}(x) e^{-N^2 S[\tilde{\rho}]} = \exp\left[-N^2 S[\rho^*] + \mathcal{O}(N)\right] \; , \; \frac{\delta S[\rho]}{\delta \tilde{\rho}(x)}\bigg|_{\tilde{\rho}=\rho^*} = 0$$

$\Longrightarrow$ integral equation for $\rho^*(x)$

$$\frac{\pi^2}{2\tilde{h}^2} x^2 - 2 \int_{-a}^{a} \rho^*(x') \ln|x - x'| \, dx' + C = 0$$
$$\int_{-a}^{a} \rho^*(x) dx = 1 \; , \; \rho^*(x) \leq 1 \; , \forall x \in [-a, a]$$

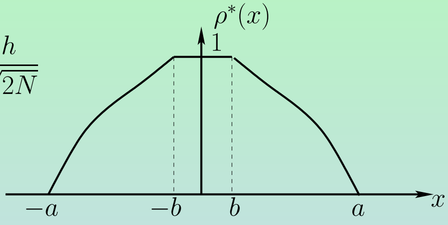
Large N asymptotics: saddle point analysis

- Analogous to the Douglas-Kazakov transition in Yang-Mills theory



$\tilde{h} > 1$

$$\tilde{h} = \frac{h}{\sqrt{2N}}$$



$\tilde{h} < 1$

Large N asymptotics: saddle point analysis

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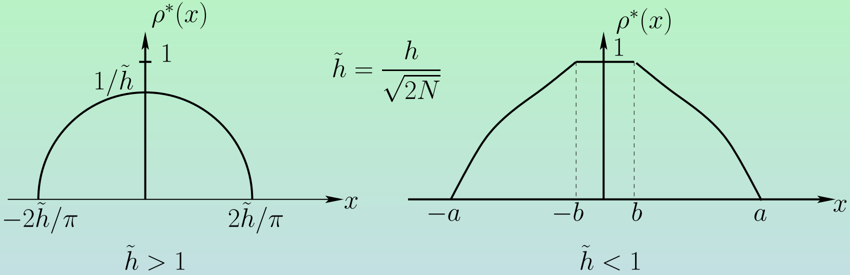

- Phase transition

$$\lim_{N \to \infty} -\frac{1}{N^2} \ln F_N(\tilde{h}\sqrt{2N}) = \begin{cases} \phi^-(\tilde{h}) \ , \ \tilde{h} < 1 \ , \\ 0 \ , \ \tilde{h} \geq 1 \end{cases}$$

Large N asymptotics: explicit expression for the left tail

$$\phi_-(h) = 2\left[F_-(\pi^2/h^2) - F_+(\pi^2/h^2)\right]$$
$$F_-(X) = -\frac{3}{4} - \frac{X}{24} - \frac{1}{2}\ln X\ ,$$
$$F_+'(X) = \frac{a^2}{6} - \frac{a^2}{12}(1-k^2) - \frac{1}{24} + \frac{a^4}{96}(1-k^2)^2 X\ ,$$

where

$$k = \frac{b}{a} \qquad (1)$$
$$a[2\mathbf{E}(k) - (1-k^2)\mathbf{K}(k)] = 1 \qquad (2)$$
$$aX = 4\mathbf{K}(k) \qquad (3)$$

where $\mathbf{E}(y)$ and $\mathbf{K}(y)$ are elliptic integrals

$$\mathbf{K}(y) = \int_0^1 \frac{dz}{\sqrt{1-y^2z^2}\sqrt{1-z^2}}\ ,\ \mathbf{E}(y) = \int_0^1 \frac{\sqrt{1-y^2z^2}}{1-z^2}\,dz$$

Large N asymptotics: third order phase transition

- Third-order phase transition

$$\lim_{N\to\infty} -\frac{1}{N^2} \ln F_N(\tilde{h}\sqrt{2N}) = \begin{cases} \phi^-(\tilde{h}) \;,\; \tilde{h} < 1 \;, \\ 0 \;,\; \tilde{h} \geq 1 \end{cases} \quad , \; \phi^-(\tilde{h}) \sim \frac{16}{3}(1-\tilde{h})^{3} \;,$$

$\Longrightarrow$ the third derivative of the partition function is discontinuous at $\tilde{h} = 1$

Large N asymptotics: third order phase transition

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$\Longrightarrow$ the third derivative of the partition function is discontinuous at $\tilde{h} = 1$

- A similar third order phase transition occurs for Gaussian β-ensembles in presence of a wall

$$Z_N(w) \propto \int_{-\infty}^{w} d\lambda_1 \cdots \int_{-\infty}^{w} d\lambda_N \exp\left[-\frac{\beta}{2}\left(N\sum_{i=1}^{N}\lambda_i^2 - \sum_{i\neq j}\ln|\lambda_i - \lambda_j|\right)\right]$$

exhibits a third order phase transition at the critical value $w_c = \sqrt{2}$

for a review see S. N. Majumdar, G. S. arXiv:1311.0580

Beyond the Coulomb gas: orthogonal polynomials

$$F_N(h) = \frac{A_N}{h^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\le j<k\le N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2h^2}\sum_{i=1}^N n_i^2}$$

Introduce discrete orthogonal polynomials

$$\sum_{n=-\infty}^{\infty} p_k(n) p_{k'}(n) e^{-\frac{\pi^2}{2h^2} n^2} = \delta_{k,k'}\, h_k \;, \; F_N(h) = \frac{\tilde{A}_N}{h^{2N^2+N}} \prod_{j=1}^{N} h_{2j-1}$$

Large N analysis of the three terms recursion relation

$$x\, p_k(x) = p_{k+1}(x) + R_k\, p_{k-1}(x) \;, \; R_k = \frac{h_k}{h_{k-1}}$$

Large N asymptotics: summary

P. J. Forrester, S. N. Majumdar, G. S. '11
G. S., S. N. Majumdar, A. Comtet, P. J. Forrester '12

$$\begin{cases} F_N(h) \sim \exp\left[-N^2\,\phi_-\left(h/\sqrt{2N}\right)\right]\,,\; h < \sqrt{2N}\ \&\ |h-\sqrt{2N}| \sim \mathcal{O}(\sqrt{N}) \\ \\ \\ \end{cases},$$

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$$\phi_+(x) = 4x\sqrt{x^2-1} - 2\ln\left[2x\left(\sqrt{x^2-1}+x\right)-1\right]$$

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$\mathcal{F}_1$ is the Tracy-Widom distribution for GOE

$$\mathcal{F}_1 = \exp\left(-\frac{1}{2}\int_t^\infty \left((s-t)\,q^2(s)+q(s)\right)\,ds\right)$$

obtained here via a double scaling analysis of the discrete OP system

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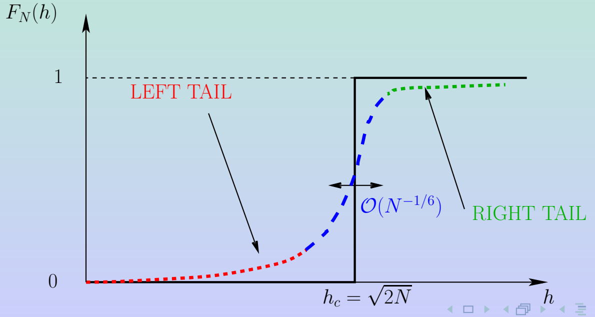
see K. Liechty '12 for a rigorous proof

Typical and large fluctuations

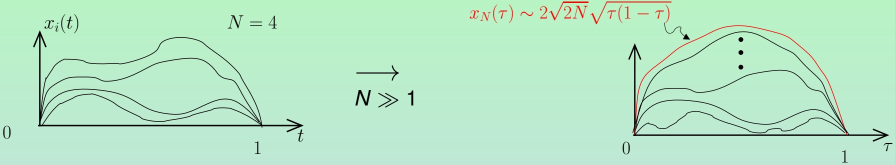
$$\begin{cases} F_N(h) & \sim \exp\left[-N^2\,\phi_-\left(h/\sqrt{2N}\right)\right],\ h<\sqrt{2N}\ \&\ |h-\sqrt{2N}|\sim\mathcal{O}(\sqrt{N}) \\ F_N(h) & \sim \mathcal{F}_1\left[2^{11/6}N^{1/6}(h-\sqrt{2N})\right],\ h\sim\sqrt{2N}\ \&\ |h-\sqrt{2N}|\sim\mathcal{O}(N^{-\frac{1}{6}}) \\ 1-F_N(h) & \sim \exp\left[-N\,\phi_+\left(h/\sqrt{2N}\right)\right],\ h>\sqrt{2N}\ \&\ |h-\sqrt{2N}|\sim\mathcal{O}(\sqrt{N}) \end{cases}$$



Outline

Non-intersecting excursions and Airy$_2$ process

- N non-intersecting excursions for large N



- Typical fluctuations of the top path are related to Airy$_2$ process minus a parabola

$$\lim_{N\to\infty} \frac{\alpha\left[x_N(\frac{1}{2} + \beta u N^{-\frac{1}{3}}) - \sqrt{2N}\right]}{N^{-\frac{1}{6}}} \stackrel{d}{=} \mathcal{A}_2(u) - u^2$$

Prähofer & Spohn '02

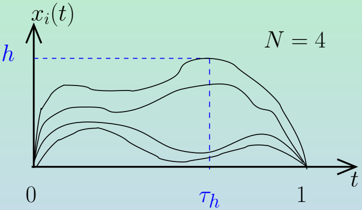
Tracy & Widom '07

$\mathcal{A}_2(u) \equiv$ Airy$_2$ process

Extreme statistics of Airy$_2$ process minus a parabola

$$\mathcal{M} = \max_{u \in \mathbb{R}} \mathcal{A}_2(u) - u^2 \,, \; \mathcal{T} = \arg\max_{u \in \mathbb{R}} \mathcal{A}_2(u) - u^2$$

Johansson '03, Corwin & Hammond '11



$$0 < x_1(t) < x_2(t) < \ldots < x_N(t)$$

$$\mathcal{H}_N = \max_{t \in [0,1]} x_N(t)$$

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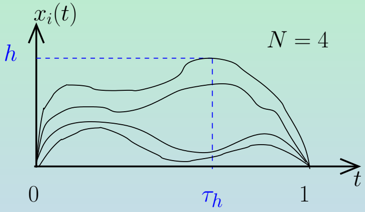

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$$\lim_{N\to\infty} 2^{2/3} N^{1/6} (\mathcal{H}_N - \sqrt{2N}) \overset{d}{=} \mathcal{M}$$

$$\lim_{N\to\infty} 2^{4/3} N^{1/3} (\mathcal{T}_{\mathcal{H}} - 1/2) \overset{d}{=} \mathcal{T}$$

Extreme statistics of Airy$_2$ process minus a parabola

$$\mathcal{M} = \max_{u \in \mathbb{R}} \mathcal{A}_2(u) - u^2 \; , \; \mathcal{T} = \arg\max_{u \in \mathbb{R}} \mathcal{A}_2(u) - u^2$$

- Our result for the PDF of the maximum on non-intersecting excursion yields

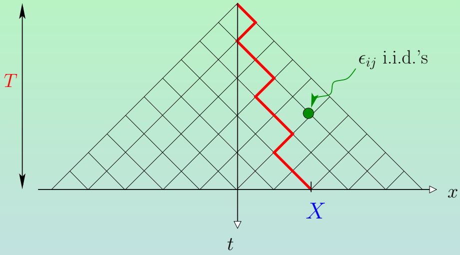
$$\mathbb{P}(\mathcal{M} \leq m) = \mathcal{F}_1(2^{2/3} m)$$

Johansson '03

Corwin, Quastel, Remenik '12

Relation to Directed Polymers in Random Media

- DPRM with one free end ("point to line")



$$\mathbb{E}(E_{\rm opt}) \sim aT \; , \; \mathbb{E}(X) = 0$$
$$E_{\rm opt} - \mathbb{E}(E_{\rm opt}) \sim \mathcal{O}(T^{1/3})$$
$$X \sim \mathcal{O}(T^{2/3})$$

Q: what is the joint pdf of $E_{\rm opt}, X$?

- $E_{\rm opt}$ $\equiv$ Energy of the optimal polymer
- X $\equiv$ Transverse coordinate of the optimal polymer

Related to KPZ growth

see review by Halpin-Healy & Zhang '95

DPRM and the Airy$_2$ process minus a parabola

- The "Airy$_2$ process minus a parabola"

$$Y(u) = \mathcal{A}_2(u) - u^2$$

where $\mathcal{A}_2(u)$ is the Airy$_2$ process   Prähofer & Spohn '02

- Fluctuations in the DPRM

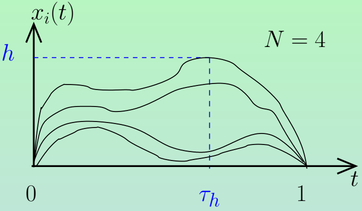
$$\lim_{T\to\infty} \frac{T^{-\frac{1}{3}}}{e_0}(E_{\rm opt}(T) - \mathbb{E}\,(E_{\rm opt}(T))) = \max_{u\in\mathbb{R}} Y(u) = \mathcal{M}$$

and

$$\lim_{T\to\infty} \frac{T^{-\frac{2}{3}}}{\xi} X = \arg\max_{u\in\mathbb{R}} Y(u) = \mathcal{T}$$

Johansson '03

Extremes of N non-intersecting excursions



$$0 < x_1(t) < x_2(t) < \ldots < x_N(t)$$

$$\mathcal{H}_N = \max_{t \in [0,1]} x_N(t)$$

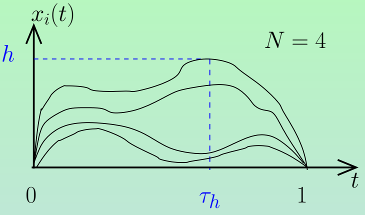
$$\mathcal{T}_{\mathcal{H}} = \arg \max_{t \in [0,1]} x_N(t)$$

$P_N(h, \tau_h) \equiv$ joint probability distribution function of $\mathcal{H}_N, \mathcal{T}_{\mathcal{H}}$

HERE :

- Compute exactly $P_N(h, \tau_h)$ for any finite N
- Typical fluctuations of $\mathcal{H}_N, \mathcal{T}_{\mathcal{H}}$ when $N \to \infty$ yield the statistics of the extremes of $\mathcal{A}_2(u) - u^2$

An exact expression for $P_N(h, \tau_h)$



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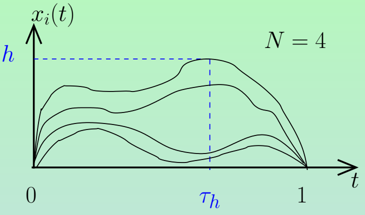
$$\mathcal{T}_{\mathcal{H}} = \arg\max_{t\in[0,1]} x_N(t)$$

$$P_N(h,\tau_h) = \frac{B_N}{h^{N(2N+1)+3}} \sum_{(n_1,\cdots,n_N,n'_N)\in\mathbb{Z}^{N+1}} \Bigg[(-1)^{n_N+n'_N}\, n_N^2 {n'_N}^2 \prod_{i=1}^{N-1} n_i^2\, \Delta_N(n_1^2,\ldots,n_{N-1}^2,n_N^2)$$

$$\times \Delta_N(n_1^2,\ldots,n_{N-1}^2,{n'_N}^2)\, e^{-\frac{\pi^2}{2h^2}\sum_{i=1}^{N-1} n_i^2 - \frac{\pi^2}{2h^2}\left[(1-\tau_h){n'_N}^2+\tau_h n_N^2\right]}\Bigg]$$

An exact expression for $P_N(h, \tau_h)$



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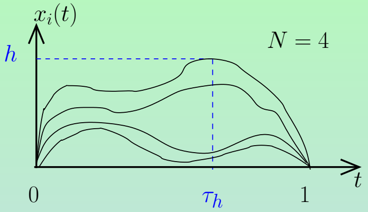
$$\left. \times \Delta_N(n_1^2,\ldots,n_{N-1}^2,{n'_N}^2)\, e^{-\frac{\pi^2}{2h^2}\sum_{i=1}^{N-1} n_i^2 - \frac{\pi^2}{2h^2}\left[(1-\tau_h){n'_N}^2+\tau_h n_N^2\right]} \right]$$

$$B_N = \frac{N\,\pi^{2N^2+N+2}}{2^{N^2+N/2+1}\prod_{j=0}^{N-1}\Gamma(2+j)\Gamma\left(\frac{3}{2}+j\right)}$$

An exact expression for $P_N(h, \tau_h)$



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Large N limit ?

Typical fluctuations for large N

$$P_N(h,\tau_h) \quad \propto \quad \sum_{(n_1,\cdots,n_N,n'_N)\in\mathbb{Z}^{N+1}} \Bigg[(-1)^{n_N+n'_N}\, n_N^2 {n'_N}^2 \prod_{i=1}^{N-1} n_i^2\, \Delta_N(n_1^2,\ldots,n_{N-1}^2,n_N^2)$$

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- Discrete orthogonal polynomials

$$\sum_{n=-\infty}^{\infty} p_k(n)p_{k'}(n)e^{-\frac{\pi^2}{2h^2}n^2} = \delta_{k,k'}\, h_k \ , \ p_k(n) = n^k + \cdots$$

Typical fluctuations for large N

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$$\sum_{n=-\infty}^{\infty} p_k(n)p_{k'}(n)e^{-\frac{\pi^2}{2h^2}n^2} = \delta_{k,k'}\, h_k \;,\; p_k(n) = n^k + \cdots$$

- After some manipulations G. S. '12

$$P_N(h,\tau_h) \propto \underbrace{\prod_{j=1}^{N} h_{2j-1}}_{\mathbb{P}[\mathcal{H}_{\mathcal{N}} \leq h] = F_N(h)} \sum_{n,m} (-1)^{n+m} n\, m \sum_{k=1}^{N} \frac{p_{2k-1}(n)p_{2k-1}(m)}{h_{2k-1}}\, e^{-\frac{\pi^2}{2h^2}\left[(1-\tau_h)n^2+\tau_h m^2\right]}$$

Typical fluctuations for large N

G. S. '12

- Large N asymptotics for extremes of excursions

$$\lim_{N\to\infty} 2^{-\frac{9}{2}} N^{-\frac{1}{2}} P_N\left(\sqrt{2N} + 2^{-\frac{11}{6}}\, s\, N^{-\frac{1}{6}}, \frac{1}{2} + 2^{-\frac{8}{3}}\, w\, N^{-\frac{1}{3}}\right) = P(s,w)$$

$$P(s,w) = \frac{4}{\pi^2}\mathcal{F}_1(s)\int_s^\infty f(x,w)f(x,-w)\,dx$$

$$f(x,w) = \int_0^\infty \zeta\Phi_2(\zeta,x)e^{-w\zeta^2}\,d\zeta$$

$$\underbrace{\frac{\partial}{\partial\zeta}\Psi = A\Psi\,,\ \frac{\partial}{\partial x}\Psi = B\Psi}_{\text{Lax Pair}}\,,\ \Psi = \begin{pmatrix}\Phi_1(\zeta,x)\\ \Phi_2(\zeta,x)\end{pmatrix}\,,\ \begin{cases}\Phi_1(\zeta,x) = \cos\left(\frac{4}{3}\zeta^3 + x\zeta\right) + \mathcal{O}(\zeta^{-1})\\ \Phi_2(\zeta,x) = -\sin\left(\frac{4}{3}\zeta^3 + x\zeta\right) + \mathcal{O}(\zeta^{-1})\end{cases}$$

Lax Pair

$$A(\zeta,x) = \begin{pmatrix} 4\zeta q & 4\zeta^2 + x + 2q^2 + 2q' \\ -4\zeta^2 - x - 2q^2 + 2q' & -4\zeta q\end{pmatrix}\,,\ B(\zeta,x) = \begin{pmatrix} q & \zeta \\ -\zeta & -q\end{pmatrix}$$

Extreme statistics of Airy$_2$ process minus a parabola

- This work: joint pdf $\hat{P}(m,t)$ of $\mathcal{M}, \mathcal{T}$ G. S. '12

$$\hat{P}(m,t) = \frac{8}{\pi^2}\mathcal{F}_1(2^{2/3}m)\int_{2^{2/3}m}^{\infty} f(x, 2^{4/3}t)f(x, -2^{4/3}t)\,dx$$

- Marginal distribution $\hat{P}(t)$ of $\mathcal{T}$ G. S. '12

$$\log \hat{P}(t) = -ct^3 + o(t^3)\;,\; t \to \infty \text{ with } c = \frac{4}{3}$$

see also Corwin & Hammond '11, Quastel & Remenik '13

Extreme statistics of Airy$_2$ process minus a parabola

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see also Corwin & Hammond '11, Quastel & Remenik '13

A more recent asymptotic analysis Bothner & Liechty '13

$$\hat{P}(t) = Ce^{-\frac{4}{3}\varphi(t)}t^{-\frac{81}{32}}(1+\mathcal{O}(t^{-\frac{3}{4}}))\ ,\ \varphi(t) = t^3 - 2t^{\frac{3}{2}} + 3t^{\frac{3}{4}}$$

Outline

Extreme statistics of Airy$_2$ process minus a parabola

$$\mathcal{M} = \max_{u\in\mathbb{R}} \mathcal{A}_2(u) - u^2 \; , \; \mathcal{T} = \arg\max_{u\in\mathbb{R}} \mathcal{A}_2(u) - u^2$$

- Our results also yield the joint pdf of $\mathcal{M}, \mathcal{T}$ as

$$\hat{P}(m,t) = 4P(2^{2/3}m, 2^{4/3}t)$$

$$P(s,w) = \frac{4}{\pi^2}\mathcal{F}_1(s)\int_s^\infty f(x,w)f(x,-w)\,dx$$

$$f(x,w) = \int_0^\infty \zeta\Phi_2(\zeta,x)e^{-w\zeta^2}\,d\zeta$$

Φ_2 is a ψ-function associated to Painlevé II

Extreme statistics of Airy$_2$ process minus a parabola

$$\mathcal{M} = \max_{u\in\mathbb{R}} \mathcal{A}_2(u) - u^2 \; , \; \mathcal{T} = \arg\max_{u\in\mathbb{R}} \mathcal{A}_2(u) - u^2$$

- A different formula was obtained first by Moreno-Flores, Quastel, Remenik arXiv:1106.2716, CMP '13

$$\hat{P}(m,t) = 2^{1/3}\mathcal{F}_1(2^{2/3}m)\int_0^\infty dx \int_0^\infty dy\, \Phi_{-t,m}(2^{1/3}x)\rho_{2^{2/3}m}(x,y)\,\Phi_{t,m}(2^{1/3}y)$$

$$\Phi_{t,m}(x) = 2e^{x\,t}[t\mathrm{Ai}(t^2+m+x) + \mathrm{Ai}'(t^2+m+x)]$$

and

$$\rho_m(x,y) = (I - \Pi_0\mathbf{B}_m\Pi_0)^{-1}(x,y)\;,\; \mathbf{B}_m(x,y) = \mathrm{Ai}(x+y+m)$$

Extreme statistics of Airy$_2$ process minus a parabola

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$$\rho_m(x,y) = (I - \Pi_0\mathbf{B}_m\Pi_0)^{-1}(x,y) \;,\; \mathbf{B}_m(x,y) = \mathrm{Ai}(x+y+m)$$

Show that the two formulas coincide !!

Solution to the Lax pair

$$\underbrace{\frac{\partial}{\partial \zeta}\Psi = A\Psi \,,\, \frac{\partial}{\partial x}\Psi = B\Psi}_{\text{Lax Pair}} \,,\; \Psi = \begin{pmatrix} \Phi_1(\zeta, x) \\ \Phi_2(\zeta, x) \end{pmatrix} \,,\, \begin{cases} \Phi_1(\zeta, x) = \cos\left(\frac{4}{3}\zeta^3 + x\zeta\right) + \mathcal{O}(\zeta^{-1}) \\ \Phi_2(\zeta, x) = -\sin\left(\frac{4}{3}\zeta^3 + x\zeta\right) + \mathcal{O}(\zeta^{-1}) \end{cases}$$

$$A(\zeta, x) = \begin{pmatrix} 4\zeta q & 4\zeta^2 + x + 2q^2 + 2q' \\ -4\zeta^2 - x - 2q^2 + 2q' & -4\zeta q \end{pmatrix} \,,\, B(\zeta, x) = \begin{pmatrix} q & \zeta \\ -\zeta & -q \end{pmatrix}$$

An explicit expression of Φ_1, Φ_2 in terms of $\mathbf{B}_m(x, y) = \mathrm{Ai}(x + y + m)$

J. Baik, K. Liechty, G. S. '12

Solution to the Lax pair

J. Baik, K. Liechty, G. S. '12

$$\mathbf{A}_s(x,y) := \mathbf{B}_s^2(x,y) = \int_0^\infty \mathrm{Ai}(x+s+\xi)\mathrm{Ai}(y+s+\xi)\,d\xi$$
$$= \frac{\mathrm{Ai}(x+s)\mathrm{Ai}'(y+s) - \mathrm{Ai}'(x+s)\mathrm{Ai}(y+s)}{x-y}$$

Define the functions Q and R as

$$Q := (\mathbf{1} - \mathbf{A}_s)^{-1}\mathbf{B}_s\delta_0\,, \qquad R := (\mathbf{1} - \mathbf{A}_s)^{-1}\mathbf{A}_s\delta_0$$

Introduce the functions

$$\Theta_1(x) := \cos\left(\frac{4}{3}\zeta^3 + (s+2x)\zeta\right)\,, \qquad \Theta_2(x) := -\sin\left(\frac{4}{3}\zeta^3 + (s+2x)\zeta\right)$$

Solution to the Lax pair

J. Baik, K. Liechty, G. S. '12

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Proposition

$$\Phi_1(\zeta,s) = \Theta_1(0) + \langle\Theta_1, R-Q\rangle_0\,, \qquad \Phi_2(\zeta,s) = \Theta_2(0) + \langle\Theta_2, R+Q\rangle_0\,,$$

where $\langle\cdot,\cdot\rangle_0$ is the inner product on $L^2[0,\infty)$

Solution to the Lax pair

J. Baik, K. Liechty, G. S. '12

Proposition

$$\Phi_1(\zeta, s) = \Theta_1(0) + \langle \Theta_1, R - Q \rangle_0 \,, \qquad \Phi_2(\zeta, s) = \Theta_2(0) + \langle \Theta_2, R + Q \rangle_0 \,,$$

- An indirect proof of this proposition was obtained by Baik '06
- Here we give a proof using the method of Tracy and Widom '94
- After further manipulations, this prop. allows to show that the two formulas for the jpdf $\hat{P}(m, t)$ do coincide

Outline

Conclusion

- Exact results for extreme statistics of N vicious walkers
- Connection with Yang-Mills theory
- Large N analysis: typical and large fluctuations
- Connection between extreme statistics of $\mathcal{A}_2(u) - u^2$ and Painlevé
- An explicit solution for the Lax pair associated to Painlevé II