IAS LECTURE FEB 2014

RANDOM MATRIX THEORY AND ZETA FUNCTIONS

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THE TEN-FOLD WAY (CARTAN)

THE 10 FAMILIES OF IRREDUCIBLE COMPACT SYMMETRIC SPACES OF E. CARTAN IN THEIR STANDARD REALIZATIONS GIVE THE RANDOM MATRIX ENSEMBLES

M. ZIRNBAUER
N. KATZ - S $\Big\}$ 1990's

RECENTLY THEY APPEAR IN THE CLASSIFICATION OF TOPOLOGICAL INSULATORS AND SUPERCONDUCTORS
SCHNYDER ET AL 2009.

$\mathcal{E}(N)$

U CUE	U(N) COMPACT N×N UNITARY MATRICES "CIRCULAR UNITARY ENSEMBLE"
O {even, odd}	ORTHOGONAL SUBGROUP N×N OF A's, $A^tA = I$,
Sp	USp(2N) SUBGROUP OF A's $A^tJA = J$, $J = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$
COE	U(N)/O(N) SYMMETRIC N×N REALIZED $B \to B^tB$
CSE	U(2N)/USp(2N), 2N×2N UNITARY H's, $J^tH^tJ = H$ IDENTIFIED BY $B \to BJB^tJ^t$
⋮	

. THE FIRST 4 ; U, SO(EVEN), SO(ODD) AND Sp, ARE THE TYPE III SYMMETRIC SPACES OF CARTAN, CORRESPOND TO THE COMPACT CLASSICAL GROUPS.

3-FOLD WAY (DYSON)

CUE, COE, CSE

THE LOCAL SCALED STATISTICAL DISTRIBUTION OF THE EIGENVALUES IN THE BULK (IE AWAY FROM ± 1) OF A TYPICAL $A \in \mathcal{E}(N)$ AS $N \to \infty$ FOLLOWS ONE OF THE ABOVE 3 LAWS.

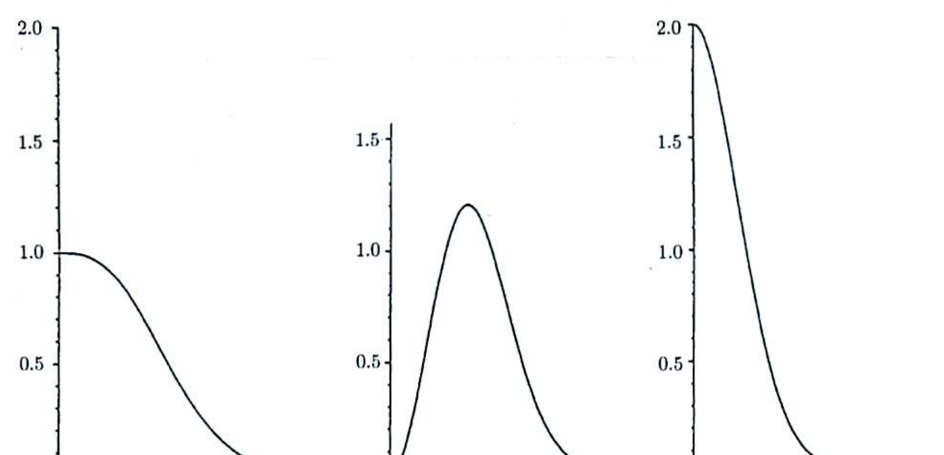
PROVED USING GAUDIN METHOD OF ORTHOGONAL POLYNOMIALS TOGETHER WITH KNOWN ASYMPTOTICS OF CLASSICAL ORTHOGONAL POLYNOMIALS

SEE:
E. DUENEZ PU THESIS C.M.P 2004

· **NB:** IN THE BULK ALL THE TYPE III ENSEMBLES BECOME CUE.

THE STATISTICS OF THE EIGENVALUES NEAR 1 FOR THE VARIOUS $\mathcal{E}(N)$'s FOLLOW CHARACTERISTIC LAWS (THAT IS AFTER SCALING AND A VARYING OVER $\mathcal{E}(N)$).

FOR EXAMPLE THE DISTRIBUTIONS OF THE EIGENVALUE NEAREST TO 1 FOR U, Sp AND SO(EVEN) ARE:



ZEROS OF ZETA FUNCTIONS:

MONTGOMERY'S PAIR CORRELATION:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1-p^{-s})^{-1}$$

$$\xi(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) \overset{\text{f.e.}}{=} \xi(1-s)$$

(RIEMANN).

WRITE THE ZEROS OF $\xi(s)$ AS

$$\rho_j = \frac{1}{2} + i\gamma_j \quad , \; \gamma_j \in \mathbb{R} \Longleftrightarrow \text{RH}$$

$$\cdots\cdots \quad < \gamma_{-1} < 0 < \gamma_1 < \gamma_2 \cdots\cdots \le \gamma_j \le \cdots$$

· LOCAL (SCALED) SPACING STATISTICS

(UNFOLD) $\tilde{\gamma}_j = \dfrac{\gamma_j \log \gamma_j}{2\pi}$, $j \geq 1$.

PAIR CORRELATION: LET $\phi \in \mathcal{S}(\mathbb{R})$ SET

$$W_2(\phi, N) = \frac{1}{N} \sum_{1 \le j \ne k \le N} \phi(\tilde{\gamma}_j - \tilde{\gamma}_k)$$

6

MONTGOMERY (73/74).

IF SUPPORT $\hat{\phi} \subset (-1,1)$ THEN

$$W_2(\phi, N) \longrightarrow \int_{-\infty}^{\infty} \phi(x)\left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx$$

AS $N \to \infty$,

AND HE CONJECTURES THAT THIS HOLDS WITHOUT ANY CONDITION ON SUPPORT $\hat{\phi}$.

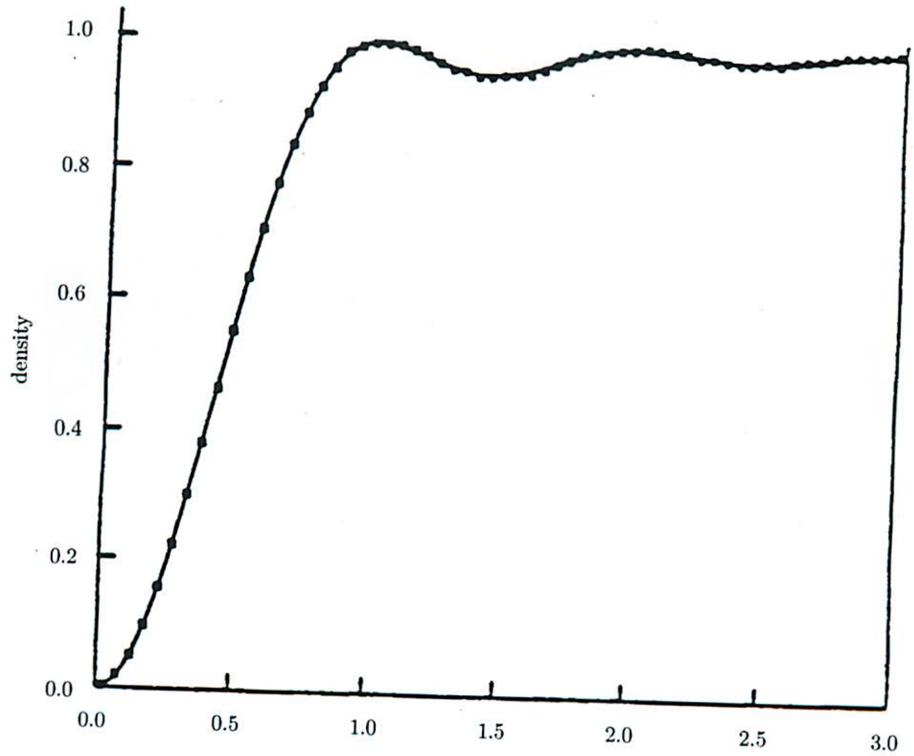


FIGURE 2. Pair correlation for zeros of zeta based on 8×10^6 zeros near the 10^{20}-th zero, versus the GUE conjectured density $1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$.

ODLYZKO

DYSON (1974) OBSERVES THAT

$$\left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right)$$

IS THE 2-POINT CORRELATION FOR CUE. SUGGESTS THAT THE LOCAL SPACING STATISTICS OF (HIGH) ZEROS OF $\zeta(s)$ FOLLOW CUE LAWS.

- CHECKED COMPREHENSIVELY NUMERICALLY BY ODLYZKO.
- TO PROVE THIS IT SUFFICES TO COMPUTE THE n-level CORRELATIONS FOR $n \geq 2$.

<u>UNIVERSALITY</u> (RUDNICK-S 90's)

LET π BE AN AUTOMORPHIC CUSP FORM ON $GL_m/\mathbb{Q}$ AND $L(s,\pi)$ ITS STANDARD L-FUNCTION (THESE GENERALIZE ZETA AND PRESUMABLY GIVE ALL L-FUNCTIONS) THEN THE n-LEVEL CORRELATIONS OF THE ZEROS OF $L(s,\pi)$ ARE GIVEN BY THE n-LEVEL DENSITIES, AT LEAST IN RESTRICTED RANGES (AND CONJECTURALLY IN ALL RANGES).

- PROOF USES THE EXPLICIT FORMULA OF RIEMANN (WEIL, GUINAND) AND CRITICALLY RANKIN-SELBERG L-FUNCTIONS.
- NUMERICALLY CONFIRMED FOR MANY π's RUMLEY, RUBINSTEIN

FAMILIES OF ZETA AND L-FUNCTIONS

EXPERIENCE SHOWS THAT UNDERSTAND AN INDIVIDUAL L-FUNCTION ONE NEEDS TO DEFORM IT IN A FAMILY.

GL_1 OR DIRICHLET L-FUNCTIONS:

$$\chi(m_1 m_2) = \chi(m_1)\chi(m_2), \quad \chi(1)=1$$

$$\chi(m+\lambda q) = \chi(m)$$

minimum period is q = "conductor of χ"

$$L(s,\chi) = \prod_p \left(1-\chi(p)p^{-s}\right)^{-1} = \sum_{m=1}^{\infty} \chi(m) m^{-s}$$

$\mathcal{F}$ = the "family" of $L(s,\chi)$'s

χ is quadratic, $\chi^2 = 1$, $\chi = \chi_d$

d is conductor
square free

Distribution of the zeros near $s = \frac{1}{2}$
"low lying"

$\rho_j = \frac{1}{2} + i\,\gamma_{j,\chi_d}$ $\quad j = \pm 1, \pm 2, \ldots$

scale $\quad \tilde{\gamma}_{j,\chi_d} = \dfrac{\gamma_{j,\chi_d} \log |d|}{2\pi}$

(NO PARAMETERS!)

- Katz-S (98): The low lying zeros of $L(s,\chi_d)$, $\chi \in \mathcal{F}$ follow the laws of eigenvalues of $USp(\infty)$ near $z = 1$.

$\Leftrightarrow$ n-level densities being those of $USp(\infty)$. That is

$\phi \in \mathcal{S}(\mathbb{R}^n)$

$$W^{(n)}(\phi, \chi_d) := \sum_{\substack{j_1, \dots, j_n \\ |j_k| \text{ distinct}}} \phi(\tilde{\gamma}_{j_1, d}, \dots, \tilde{\gamma}_{j_n, d})$$

as $X \to \infty$

$$\frac{1}{|\{\chi_d : |d| \leq X\}|} \sum_{|d| \leq X} W^{(n)}(\phi, \chi_d) \to \int_{\mathbb{R}^n} \phi(x) W^{(n)}_{USp}(x)\, dx \quad \text{——} (*)$$

where $W^{(n)}_{USp}(x) = \det \left(K_{USp}(x_i, x_j) \right)_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$

$$K_{USp}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)} - \frac{\sin \pi(x+y)}{\pi(x+y)}.$$

The n-level densities have been computed for

$$\text{support } \hat{\Phi} \subset \sum_{j=1}^{n} |\xi_j| < 1$$

(RUBINSTEIN)

$$\text{support } \hat{\Phi} \subset \sum_{j=1}^{n} |\xi_j| < 2 \quad \text{(GAO)}$$

(but cannot identify the answer with (*)).

NUMERICAL CONFIRMATION OF ALL SOME STATISTICS FOR THIS AND OTHER FAMILIES – RUBINSTEIN.

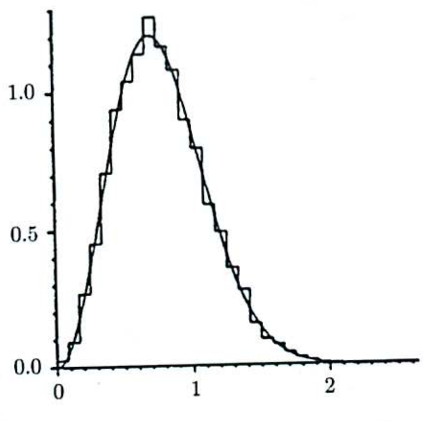


FIGURE 7. 1st zero above 0 for $L(s, \chi_d)$, $10^{12} < |d| < 10^{12} + 200000$,

vs. $\nu_1(Sp)$.

(12)

The basis for understanding these phenomena - that is a symmetry type associated with $\mathcal{F}$ and the universal CUE law comes from function field considerations (Katz-S).

- Replace $\mathbb{Q}$ by $\mathbb{F}_q(t)$

QUADRATIC FAMILY:

K varies over quadratic extentions of $\mathbb{F}_q(t)$, $T = q^{-s}$

$$\zeta_K(T) = \frac{P(T,K)}{(1-T)(1-qT)}$$

P is a polynomial of degree $2g$ where g is the genus of K. Its zeros are on the circle

$$\rho = q^{-1/2} e^{i\theta} \qquad (RH)$$

and asks for the distribution of the θ's as $g(K) \to \infty$.

Using techniques from monodromy groups of families of varieties (in this example curves) over finite fields and their scaling limits (specifically GAUDIN's methods) one shows that as $q \to \infty$ and $g \to \infty$

.) universally in the bulk (i.e over all the zeros) the scaling limits of these distributions for almost all members of ANY family is CUE.

..) The distribution near the central point ($z = 1$) is one of the 4 symmetry types (type III spaces)

$$U(\infty), \ Sp(\infty), \ SO_{even}(\infty), SO_{odd}(\infty)$$

Back to $\mathbb{Q}$

A definition of a family $\mathcal{F}$ of automorphic forms π on GL_m and hence of L-functions $L(s,\pi)$ is given in (Shin-Templier-S 2014).

There are two sources for forming $\mathcal{F}$:

(1) <u>HARMONIC:</u> The forms π are defined through spectral theory of arithmetic locally symmetric spaces and transfered to GL_m by functoriality.

(2) <u>ALGEBRAIC</u>: V_t, $t \in \mathbb{P}^n$

A FAMILY OF (SMOOTH) PROJECTIVE VARIETIES DEFINED OVER $\mathbb{Q}$.
$L(s, V_t)$ ($= L(s, \pi_t)$) THE HASSE-WEIL ZETA FUNCTION ON A PIECE OF COHOMOLOGY.

Given such $\mathcal{F}$, $L(s,\pi)$ $\pi \in \mathcal{F}$ each π has an "analytic conductor" $c(\pi)$ (Iwaniec-S) which measures the 'height' of π and also the density of zeros near $s = 1/2$ (i.e. normalize by $\frac{\log c(\pi)}{2\pi}$).

$$\mathcal{F}_X = \{\pi \in \mathcal{F} : c(\pi) \leq X\} \text{ is finite.}$$

to compute the n-level densities for

$$L(s,\pi) = \sum_{n_1=1}^{\infty} \frac{\lambda_\pi(n_1)}{n_1^s}$$

one needs at least the bahavior of

$$\frac{1}{|\mathcal{F}_X|} \sum_{\pi \in \mathcal{F}_X} \lambda_\pi(t) \quad \text{as } X \to \infty$$

for each t (and some uniformity).

(·) For the HARMONIC families this is achieved through the trace formula

(··) For the algebraic families one uses techniques from monodromy groups of families (Grothendieck, Deligne, Katz).

(···) These are input into the explicit formula of Riemann-Guinand-Weil.

There are works by many people in the last 14 years studying low lying zeros for various families, all are special cases of the general formation. Technically once the n-level densities are computed an important issue is the size of the support of $\hat{\phi}$.

TWO INTERESTING RECENT DEVELOPMENTS:

1) ENTIN, RODITTY, RUNDICK (2012)

THE SHOW THAT FOR THE ORIGINAL QUADRATIC FAMILY, $\chi^2 = 1$, THE n-LEVEL DENSITIES ARE

$W^{(n)}_{USp}$ FOR support $\hat{\phi} \subset \left(\sum |\xi_j| < 2 \right)$

This involves establing an infinite set of complicated combinatorial identities. Their proof is similar to that of the "FUNDAMENTAL LEMMA".

They examine the n-level densities for K a quadratic extention of $\mathbb{F}_p(t)$, p large, directly with the analogue of the above support condition on $\hat{\phi}$. They show that the combinatorial identity needed is the same as that for χ^2 and $\mathbb{Q}$. Then averaging over p and using Katz-S they infer the combinatorial identities.

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2) SHIN-TEMPLIER CONSIDER THE FOLLOWING GENERAL FAMILY (HARMONIC)

G a reductive algebraic group $/\mathbb{Q}$

${}^LG$ its Langlands dual group

$\rho : {}^LG \longrightarrow GL_m(\mathbb{C})$ irreducible.

Assume that $G(\mathbb{R})$ carries discrete series and restrict to automorphic representations of $G(\mathbb{A})$ for which π_∞ is discrete series and either their weights or levels or both increase.

The family $\mathcal{F}$ in question are the L-functions

$L(s, \pi, \rho)$, π as above.

They compute the 1-level density for this family and show that the symmetry type of $\mathcal{F}$ is

$$U(\infty),\ Sp(\infty),\ O(\infty)$$

according as the Frobenius-Schur indicator of ρ is 0 (i.e. no invariant pairing) 1 (i.e ρ has a symmetric pairing) and -1 (ρ has an alternating pairing).

FOR OUR GENERAL $\mathcal{F}$'s ONE CAN COMPUTE (FINITE DIMENSIONAL) SATO–TATE GROUPS $H_{ST}(\mathcal{F}) \subset GL_m(\mathbb{C})$ ASSOCIATED WITH THE DISTRIBUTION (VERTICAL) OF $\lambda_\pi(t)$, $\pi \in \mathcal{F}$.

VARIOUS INDICATORS ASSOCIATED WITH $H_{ST}(\mathcal{F})$ THE DETERMINE THE SYMMETRY TYPE OF $\mathcal{F}$.

$\Rightarrow$ ONLY THE 4 ENSEMBLES CORRESPONDING TO TYPE $\underline{\text{III}}$ SYMMETRIC SPACES ARISE. SO AT LEAST FOR THE THEORY OF ZETA FUNCTIONS IT APPEARS THAT THESE ARE THE ONLY RELEVENT ONES.

THERE ARE MANY APPLICATIONS OF THE DISTRIBUTION OF THE ZEROS AND OF THE SYMMETRY TYPES FOR $\mathfrak{F}$'S.

SOME ARE:

- MOMENTS OF ZETA AND L-FUNCTIONS (KEATING-SNAITH)

- NONVANISHING OF L-FUNCTIONS AT CENTRAL POINTS, MORDELL-WEIL RANKS VIA BSD.

- SUBCONVEXITY AND APPROXIMATIONS TO LINDELOF AND RIEMANN.

AT A MORE SPECULATIVE LEVEL THE SYMMETRY IS PERHAPS, AS IN THE FUNCTION FIELD, CONNECTED TO A NONDEGENERATE PAIRING PRESERVED IN A SPECTRAL INTERPRETATION OF THE ZEROS.