# Concentration inequalities for linear cocycles and their applications to problems in dynamics and mathematical physics*

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## The notion of a linear cocycle

Given

an ergodic system $(X, \mu, T)$;

a measurable function $A : X \to \mathrm{Mat}(m, \mathbb{R})$,

we call linear cocycle the skew-product map $F : X \times \mathbb{R}^m \to X \times \mathbb{R}^m$ defined by

$$F(x, v) = (Tx,\, A(x)v)\,.$$

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This map defines a new dynamical system on the bundled space $X \times \mathbb{R}^m$, and its iterates are $F^n(x, v) = (T^n x,\, A^{(n)}(x)v)$, where

$$A^{(n)}(x) := A(T^{n-1}x) \cdot \ldots \cdot A(Tx) \cdot A(x)\,.$$

We usually fix the base dynamics $T$ and identify the cocycle $F$ with the function $A$ defining its fiber action.

## Example: random (i.i.d.) cocycles

- Base dynamics: $(X, \mu, T)$ is a Bernoulli shift i.e. given a probability space of symbols $(\Sigma, \nu)$, we put

  $X = \Sigma^{\mathbb{Z}}$,
  $\mu = \nu^{\mathbb{Z}}$ and
  if $x = \{x_k\}_{k\in\mathbb{Z}} \in X$ then $Tx = \{x_{k+1}\}_{k\in\mathbb{Z}}$.

- Fiber action: $A\colon X \to \mathrm{GL}(m, \mathbb{R})$ locally constant i.e. it only depends on the zeroth coordinate:
  If $x = \{x_k\}_{k\in\mathbb{Z}}$, then $A(x) = \widehat{A}(x_0)$ for some measurable map $\widehat{A}\colon \Sigma \to \mathrm{GL}(m, R)$.

**Related example:** Markov cocycles.

## Example: quasi-periodic cocycles

- Base dynamics: $(X, \mu, T)$ is a torus translation
  $X = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ ($d = 1$ or $d > 1$),
  $\mu =$ Haar measure and
  $Tx = x + \omega$ for some ergodic translation $\omega \in \mathbb{T}^d$.

- Fiber action: $A \colon \mathbb{T}^d \to \mathrm{Mat}(m, \mathbb{R})$ real analytic
  hence holomorphic in a neighborhood of the torus.

**Other examples:** cocycles over a skew-translation or over the doubling map or over a hyperbolic toral automorphism.

## The Schrödinger cocycle

Given an ergodic system $(X, \mu, T)$, an observable $f\colon X \to \mathbb{R}$ determines the one-parameter family of linear cocycles $(T, A_E)$, where $A_E\colon X \to \mathrm{SL}(2, \mathbb{R})$,

$$A_E(x) := \begin{bmatrix} f(x) - E & -1 \\ 1 & 0 \end{bmatrix}.$$

$A_E$ is called a Schrödinger cocycle as it is related to the discrete Schrödinger operator with dynamically defined potential $H(x)\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

$$[H(x)\psi]_n = -(\psi_{n+1} + \psi_{n-1}) + f(T^n x)\,\psi_n$$

and to the corresponding Schrödinger equation

$$[H(x)\psi = E\,\psi\,.$$

## The Lyapunov exponents of a linear cocycle

The singular values of a matrix $g \in \mathrm{Mat}(m, \mathbb{R})$ are denoted by

$$s_1(g) \geqslant s_2(g) \geqslant \ldots \geqslant s_m(g) \geqslant 0 \,.$$

Let $(X, \mu, T)$ be an ergodic base dynamical system and let $A \colon X \to \mathrm{Mat}(m, \mathbb{R})$ be an integrable cocycle, i.e.

$$\log^+ \|A\| \in L^1(X, \mu) \,.$$

By Furstenberg-Kesten's theorem, for every $1 \leqslant j \leqslant m$, the following limit exists:

$$L_j(A) := \lim_{n\to\infty} \frac{1}{n} \log s_j(A^{(n)}(x)) \quad \mu \text{ a.e. } x \in X$$

and it is called the $j$-th Lyapunov exponent of $A$.

## The maximal Lyapunov exponent

Given:

an ergodic dynamical system $(X, \mu, T)$,

and a matrix-valued absolutely integrable observable $A\colon X \to \mathrm{Mat}(m, \mathbb{R})$,

consider the $n$-th iterate of $A$

$$A^{(n)}(x) := A(T^{n-1}x) \ldots A(Tx)\, A(x).$$

Then for $\mu$ a.e. $x \in X$, the “geometric average”

$$\frac{1}{n} \log\|A^{(n)}(x)\| \to L_1(A) \quad \text{as } n \to \infty,$$

and $L(A) := L_1(A)$ is the maximal Lyapunov exponent of the cocycle $A$.

# Concentration inequalities in classical probabilities

## The large deviation principle of Cramér:

Let $\xi_0, \xi_1, \ldots$ be a real valued i.i.d. random process, and let $S_n = \xi_0 + \xi_1 + \ldots + \xi_{n-1}$ be its sum process. Assuming finite exponential moments, asymptotically,

$$\mathbb{P}\left[\,\left|\frac{1}{n}\,S_n - \mathbb{E}\xi_0\right| > \varepsilon\,\right] \approx e^{-I(\varepsilon)\,n}$$

where $I(\varepsilon)$ is an explicit rate function.

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**Hoeffding's inequality:** Let $\xi_0, \xi_1, \ldots, \xi_{n-1}$ be a real valued independent random process and let $S_n := \xi_0 + \xi_1 + \ldots + \xi_{n-1}$ be its sum. If for some finite constant $C$, $\left|\xi_i\right| \leqslant C$ a.s. for all $i = 0, \ldots, n-1$, then

$$\mathbb{P}\left[\,\left|\frac{1}{n}\,S_n - \mathbb{E}\big(\frac{1}{n}\,S_n\big)\right| > \epsilon\,\right] \leqslant 2\,e^{-(2C)^{-2}\,\epsilon^2\,n}\,.$$

## Concentration inequalities in dynamical systems

Given an ergodic system $(X, \mu, T)$ and an integrable observable $\xi\colon X \to \mathbb{R}$, the sequence of random variables

$$\xi_k := \xi \circ T^k \quad k \geqslant 0$$

is a real-valued stationary process.

Then the ergodic sums

$$S_n\,\xi(x) := \xi(x) + \xi(Tx) + \ldots + \xi(T^{n-1}x)$$

represent the corresponding sum process.

## Concentration inequalities in dynamical systems

Hoeffding-type inequalities are available for a wide class of (non-uniformly) hyperbolic dynamical systems and for a large space of observables.

**Theorem.** (J.-R. Chazottes and S. Gouëzel 2014) Let $(X, \mu, T)$ be a dynamical system modeled by a Young tower with exponential tails.
There is a constant $C > 0$ such that for every Lipschitz observable $\xi$ and for every $n \in \mathbb{N}$,

$$\mu\left\{ x \in X \colon \left| \frac{1}{n} S_n \xi(x) - \int_X \xi d\mu \right| > \epsilon \right\} \leqslant 2\, e^{-(2C\,\mathrm{Lip}(\xi))^{-2}\, \epsilon^2\, n}.$$

*Reference:* J.-R. Chazottes, Fluctuations of Observables in Dynamical Systems: From Limit Theorems to Concentration Inequalities, 2015.

## Concentration inequalities in dynamical systems

Such estimates are also available for the torus translation $T\colon \mathbb{T} \to \mathbb{T}$, $Tx = x + \omega$ with observables of quite low regularity, provided $\omega$ satisfies a generic Diophantine condition (DC).

**Theorem.** Let $u\colon \mathcal{A}_r \to [-\infty, \infty)$ be a subharmonic function on the annulus $\mathcal{A}_r$ of width $r$ around the torus $\mathbb{T}$. Let $\xi$ be the restriction of $u$ to the torus $\mathbb{T}$. Then

$$\mu \left\{ x \in \mathbb{T}\colon \left| \frac{1}{n} S_n \xi(x) - \int_{\mathbb{T}} \xi \, d\mu \right| > \epsilon \right\} \leqslant C e^{-c \epsilon n},$$

for constants $c > 0$ and $C < \infty$ that depend on certain uniform measurements of $u$ and on the DC of $\omega$.

*Reference:* This result and versions thereof were obtained by

Bourgain and Goldstein; Goldstein and Schlag; K.; Duarte and K.

## Concentration inequalities for iterates of linear cocycles

Let $(X, \mu, T)$ be an ergodic base dynamical system.

Let $A \colon X \to \mathrm{Mat}(m, \mathbb{R})$ be a linear cocycle over $T$. Recall the notation

$$A^{(n)}(x) := A(T^{n-1}x)\, \ldots\, A(Tx)\, A(x)\,.$$

We say that the cocycle $A$ satisfies an LDT estimate if

$$\mu \left\{ x \in X \colon \left| \frac{1}{n} \log \| A^{(n)}(x) \| - L^{(n)}(A) \right| > \epsilon \right\} < C\, e^{-c\, \epsilon^2\, n}$$

for some constants $c > 0$ and $C < \infty$ and for all $n \in \mathbb{N}$.

The LDT is called uniform if the constants $c, C$ are stable under small perturbations of $A$ in some given metric space of cocycles.

## Concentration inequalities for iterates of linear cocycles

**Problem.** Establish uniform LDT estimates (and other types of statistical properties) for various spaces of linear cocycles, i.e. for appropriate

base dynamics $(X, \mu, T)$;

regularity assumptions on the cocycle $A$;

topology on the space of cocycles.

Surprisingly few such results are available.

## Some available results

### Random cocycles.

- Limit theorems: Guivarc'h, Raugi, Le Page, Bougerol (90s).
- Concentration inequalities: Duarte and K. (2015). The results above hold under an irreducibility assumption.
- Irreducibility assumption eliminated in dim 2: Duarte and K. (2018).

### Quasi-periodic cocycles.

Assuming the translation frequency is Diophantine, concentration inequalities were obtained for:

- Real-analytic cocycles: Bourgain and Goldstein; Goldstein and Schlag; Duarte and K.
- Non-analytic cocycles: K.; Wang and Zhang.

## Motivation regarding such concentration inequalities

They are relevant in the study of the spectral properties of discrete Schrödinger type operators in mathematical physics.

Have consequences on the Lyapunov exponents of linear cocycles, namely on their positivity, simplicity, continuity properties.

## A general approach to proving continuity properties of Lyapunov exponents

We devised an abstract scheme to prove quantitative continuity of the Lyapunov exponents, one that is applicable to any base dynamics, provided that uniform LDT estimates are available in the given space of cocycles.

This scheme relies upon ideas introduced in the context of quasi-periodic Schrödinger cocycles in:

*Reference:* Goldstein and Schlag, Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions, Annals of Math, 2001.

## The abstract continuity theorem (Duarte and K.)

Let $(X, \mu, T)$ be an ergodic system and let $(\mathcal{C}, d)$ be a metric space of $SL(2, \mathbb{R})$-valued cocycles over it.

We assume the following:

- ◇ $\|A\| \in L^\infty(X, \mu)$ for all $A \in \mathcal{C}$.
- ◇ $d(A, B) \geqslant \|A - B\|_{L^\infty}$ for all $A, B \in \mathcal{C}$.
- ◇ Every cocycle $A \in \mathcal{C}$ with $L(A) > 0$ satisfies a uniform LDT estimate.

Then the following statements hold.

- The Lyapunov exponent $L\colon \mathcal{C} \to \mathbb{R}$ is a continuous function.
- Locally near every cocycle $A \in \mathcal{C}$ with $L(A) > 0$, the Lyapunov exponent is Hölder continuous.

## Continuity of LE for random cocycles

- ◇ Ruelle (1979)
- ◇ Furstenberg and Kifer (1983)
- ◇ Le Page (1989)
- ◇ Peres (1991)
- ◇ Bocker and Viana (2010)
- ◇ Ávila, Eskin and Viana (2014 + work in progress)
- ◇ Malheiro and Viana (2015)
- ◇ Backes, Brown, Butler (2015)
- ◇ Duarte and K. (2015 + work in progress)

## Continuity of LE for quasi-periodic cocycles

- ◇ Goldstein and Schlag (2001)
- ◇ Bourgain and Jitomirskaya (2001)
- ◇ Bourgain (2005)
- ◇ Marx and Jitomirskaya (2012)
- ◇ Ávila, Jitomirskaya and Sadel (2014)
- ◇ K. (2005; 2014)
- ◇ Wang and Zhang (2015)
- ◇ Duarte and K. (2014; 2015; 2016)