Feynman categories

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References

Main paper(s)

with B. Ward and J. Zuniga.

1. *Feynman categories: Arxiv 1312.1269*
2. Examples in: *The odd origin of Gerstenhaber, BV and the master equation* arXiv:1208.3266

Goals

There are two main goals

1. Provide a *lingua universalis* for operations and relations which includes all known such gadgets as examples.
2. Do universal constructions in general.

Applications

Find out information of objects with operations. E.g. Gromov-Witten invariants, String Topology, etc.

Plan

Warm up I

Operations and relations for Associative Algebras

- Data: An object A and a multiplication $\mu : A \otimes A \to A$
- An associativity equation $(ab)c = a(bc)$.
- Think of μ as a 2-linear map. Let $\circ_1$ and $\circ_2$ be substitution in the 1st resp. 2nd variable: The associativity becomes
$$\boxed{\mu \circ_1 \mu = \mu \circ_2 \mu : A \otimes A \otimes A \to A}.$$
$$\mu \circ_1 \mu(a, b, c) = \mu(\mu(a, b), c) = (ab)c$$
$$\mu \circ_2 \mu(a, b, c) = \mu(a, \mu(b, c)) = a(bc)$$
- We get n–linear functions by iterating μ:
$a_1 \otimes \cdots \otimes a_n \to a_1 \dots a_n$.
- There is a permutation action $\tau\mu(a, b) = \mu \circ \tau(a, b) = ba$
- This give a permutation action on the iterates of μ. It is a free action there and there are $n!$ n–linear morphisms generated by μ and the transposition.

Warm up II

Categorical formulation for representations of a group G.

- $\underline{G}$ the category with one object $*$ and morphism set G.
- $f \circ g := fg$.
- This is associative ✓
- Inverses are an extra structure $\Rightarrow$ $\underline{G}$ is a groupoid.
- A representation is a functor ρ from $\underline{G}$ to $\mathcal{V}ect$.
- $\rho(*) = V$, $\rho(g) \in Aut(V)$
- Induction and restriction now are pull–back and push–forward (Lan) along functors $\underline{H} \to \underline{G}$.

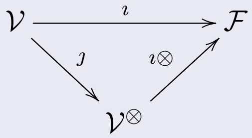
Feynman categories

Data

1. $\mathcal{V}$ a groupoid
2. $\mathcal{F}$ a symmetric monoidal category
3. $\imath : \mathcal{V} \to \mathcal{F}$ a functor.

Notation

$\mathcal{V}^{\otimes}$ the free symmetric category on $\mathcal{V}$ (words in $\mathcal{V}$).



Feynman category

Definition

A triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ of objects as above is called a Feynman category if

1. The monoidal functor $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
2. Fix $\phi : X \to X'$, $X' \simeq \bigotimes_{v\in I} \imath(*_v)$: there are $X_v \in \mathcal{F}$, and $\phi_v \in Hom(X_v, \imath(*_v))$ s.t. these fit into a commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{\quad\phi\quad} & X' \\ {\scriptstyle\simeq}\big\downarrow & & \big\downarrow{\scriptstyle\simeq} \\ \bigotimes_{v\in I} X_v & \xrightarrow{\bigotimes_{v\in I}\phi_v} & \bigotimes_{v\in I} \imath(*_v) \end{array} \tag{1}$$

3. For any $* \in \mathcal{V}$, $(\mathcal{F} \downarrow *)$ is essentially small.

Example 1

$\mathcal{F} = Sur$, $\mathcal{V} = \mathbb{I}$

- Sur be the category of finite sets and surjection with $\amalg$ as monoidal structure
- $\mathbb{I}$ be the trivial category with one object $*$ and one morphism id_*.
- $\mathbb{I}^{\otimes}$ is equivalent to the category with objects $\bar{n} \in \mathbb{N}_0$ and $Hom(\bar{n}, \bar{n}) \simeq \mathbb{S}_n$, where we think $\bar{n} = \{1, \ldots, n\} = \{1\} \amalg \cdots \amalg \{1\}$, $1 = \imath(*)$.
- $\mathbb{I}^{\otimes} \simeq Iso(Sur)$ ✓.
- $T \simeq \{1, \ldots, n\}$.

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ \Big\downarrow \simeq & & \Big\downarrow \simeq \\ \amalg_{i=1}^{|T|} f^{-1}(i) & \xrightarrow{\amalg f|_{f^{-1}(i)}} & \amalg_{i=1}^{|T|} \imath(*) \end{array}$$

$\mathcal{O}ps$ and $\mathcal{M}ods$

Definition

Fix a symmetric monoidal category $\mathcal{C}$ and $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ a Feynman category.

- Consider the category of strong symmetric monoidal functors $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F}, \mathcal{C})$ which we will call $\mathcal{F}$–ops in $\mathcal{C}$
- $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V}, \mathcal{C})$ will be called $\mathcal{V}$-modules in $\mathcal{C}$ with elements being called a $\mathcal{V}$–mod in $\mathcal{C}$.

Theorem

The forgetful functor $G : \mathcal{O}ps \to \mathcal{M}ods$ has a right adjoint F (free functor) and this adjunction is monadic.

Example 2

The Borisov-Manin category of graphs.

1. A graph Γ is a tuple $(F, V, \partial, \imath)$ of flags F, vertices V, incidence $\partial : F \to V$ and flag gluing $\imath : F^{\circlearrowleft}$. $\imath^2 = id$. We either glue two half-edges or keep a tail.
2. A graph morphism $\phi : \Gamma \to \Gamma'$ is a triple $(\phi_V, \phi^F, \imath_\phi)$, where $\phi_V : V \to V'$ is a surjection on vertices, $\phi^F : F' \to F$ is an injection and $\imath_\phi : F \setminus \phi^F(F')^{\circlearrowleft}$ a pairing (ghost edges).
3. A graph morphism from a collection of corollas Γ to a corolla $*$ has a ghost graph $\mathbb{\Gamma} = (V_\Gamma, F_\Gamma, \imath_\phi)$

$\mathfrak{F} = (\mathcal{A}gg, \mathcal{C}rl, \imath)$

$\mathcal{A}gg$ the full subcategory whose objects are aggregates of corollas.
$\mathcal{C}rl$ the category of corollas with isomorphisms.

Examples

$\mathcal{O}ps$

We can restrict the underlying ghost graphs of maps to corollas to obtain several Feynman categories. The $\mathcal{O}ps$ will then yield types of operads or operad like objects.

Types of operads and graphs

$\mathcal{O}ps$	Graphs
Operads	rooted trees
Cyclic operads	trees
Modular operads	connected graphs (add genus marking)
PROPs	directed graphs (and input output marking)
NC modular operad	graphs (and genus marking)
...	...

Algebras

Standard structures on an Object of $\mathcal{C}$

To define algebras, one needs a standard functor $\mathcal{O}(X)$ for each object $X \in \mathcal{C}$. Usually $\mathcal{C}$ with internal Hom compatible with monoidal structure.

Example for operads: $\mathcal{O}(X)(*_n) = Hom(X^{\otimes n}, X)$

Algebras

X is an algebra over $\mathcal{O}$ if there is a natural transformation $\mathcal{O} \to \mathcal{O}(X)$.

THINK OPERATIONS ON X.

Further examples

Enriched version

We can consider Feynman categories and target categories enriched over another monoidal category, such as $\mathcal{T}op$, $\mathcal{A}b$ or $dg\mathcal{V}ect$.

Theorem

The category of Feynman categories with trivial $\mathcal{V}$ enriched over $\mathcal{E}$ is equivalent to the category of operads in $\mathcal{E}$ with the correspondence given by $O(n) :=: Hom(\bar{n}, \bar{1})$. The $\mathcal{O}ps$ are now algebras over the underlying operad.

More

Other examples are twisted modular operads, non–sigma versions, the simplicial category, crossed simplicial groups, FI–algebras.

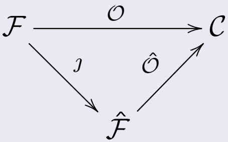
Universal constructions: What we can do:

1. Push–forwards and pull–backs along functors between Feynman categories.
   THINK INDUCTION/RESTRICTION/EXTENSION BY 0.
2. Co(bar) transforms and resolutions. Think (co)bar transformation/resolution for algebras as well as Feynman transforms and master equations.
   NB: THIS NEEDS MODEL CATEGORY THEORY WHICH WE PROVIDE
3. Universal operations. Lie–brackets, BV etc.
4. Hopf algebra structures (joint with I. Gálvez–Carrillo and A. Tonks).
   This includes Connes–Kreimers Renormalization Hopf algebra, Goncharov's Hopf algebra for multi–zetas (polylogs) and Baues' double cobar Hopf algebra.

Universal operations

Cocompletion

Let $\hat{\mathcal{F}}$ be the cocompletion of $\mathcal{F}$. This is monoidal with Day convolution $\circledast$. If $\mathcal{C}$ is cocomplete, and $\mathcal{O} \in \mathcal{O}ps$ factors.



Theorem

Let $\mathbb{I} := \mathrm{colim}_{\mathcal{V}} \jmath \circ \imath \in \hat{\mathcal{F}}$ *and let* $\mathcal{F}_{\mathcal{V}}$ *the symmetric monoidal subcategory generated by* $\mathbb{I}$*. Then* $\mathfrak{F}_{\mathcal{V}} := (\mathcal{F}_{\mathcal{V}}, \mathbb{I}, \imath_{\mathcal{V}})$ *is a Feynman category. (This gives an underlying operad of universal operations).*

Examples

Operads

$\mathfrak{O}$ the Feynman category for operads, $\mathcal{C} = dg\mathcal{V}ect$.

- Then $\hat{\mathcal{O}}(\mathbb{I}) = \bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$ and the Feynman category is (weakly) generated by $\circ := [\sum \circ_i]$. (This is a two line calculation).
- This gives rise to the Lie bracket by using the anti–commutator. The operations go back to Gerstenhaber and Kapranov-Manin.
- It lifts to the non-Sigma case i.e. a pre–Lie structure on $\bigoplus_n \mathcal{O}(n)_{\mathbb{S}_n}$.

Universal Operations

$\mathfrak{F}$	Feynman category for	$\mathfrak{F}, \mathfrak{F}_{\mathcal{V}}, \mathfrak{F}^{nt}_{\mathcal{V}}$	weakly gen. subcat.
$\mathfrak{O}$	Operads	rooted trees	$\mathfrak{F}_{pre-Lie}$
$\mathfrak{O}^{odd}$	odd operads	rooted trees + orientation of set of edges	odd pre-Lie
$\mathfrak{O}^{pl}$	non-Sigma operads	planar rooted trees	all $\circ_i$ operations
$\mathfrak{O}_{mult}$	Operads with mult.	b/w rooted trees	pre-Lie + mult.
$\mathfrak{C}$	cyclic operads	trees	commutative mult.
$\mathfrak{C}^{odd}$	odd cyclic operads	trees + orientation of set of edges	odd Lie
$\mathfrak{M}^{odd}$	$\mathfrak{K}$–modular	connected + orientation on set of edges	odd dg Lie
$\mathfrak{M}^{nc,odd}$	nc $\mathfrak{K}$-modular	orientation on set of edges	BV

Table: Here $\mathfrak{F}_{\mathcal{V}}$ and $\mathfrak{F}^{nt}_{\mathcal{V}}$ are given as $\mathcal{F}_{\mathcal{O}}$ for the insertion operad. The former for the type of graph with unlabelled tails and the latter for the version with no tails.

Examples

Odd/anti–cyclic Operad

The universal operations are (weakly) generated by a Lie bracket. $[\ ,\ ] := [\sum_{st} \circ_{st}]$, (see [KWZ]). This actually lifts to cyclic coinvariants (non–sigma cyclic operads).
Specific examples:

- $End(V)$ for a symplectic vector space is anti–cyclic.
- Any tensor product: $(\mathcal{O} \otimes \mathcal{P})(n) := \mathcal{O}(n) \otimes \mathcal{P}(n)$ with $\mathcal{O}$ cyclic and $\mathcal{P}$ anti–cyclic is anti–cyclic.

Three geometries (Kotsevich, Conant-Vogtmann

Fix V^n n–dim symplectic $V^n \to V^{n+1}$. For each n get Lie algebras
(1) $Comm \otimes End(V^n)$ (2) $Lie \otimes End(V^n)$ (3) $Assoc \otimes End(V^n)$
Take the limit as $n \to \infty$.

Odd versions

Odd versions

Given a well-behaved presentation of a Feynman category (generators+relations for the morphisms) we can define an odd version which is enriched over $\mathcal{A}b$.

Odd Feynman categories over graphs

In the case of underlying graphs for morphisms, odd usually means that edges get degree 1, that is we use a Kozsul sign with that degree.

(Co)Bar Feynman transform

Algebra case

- C associative co–algebra. $\Omega C := Free_{alg}(\Sigma^{-1}\bar{C})+$ differential coming from co–algebra structure
- A associative algebra. $BA = T\Sigma^{-1}\bar{A}$ + co–differential from algebra structure
- ΩBA is a free resolution.
- A say finite dim or graded with finite dim pieces $\check{A}$ its dual. $FA := \Omega\check{A}$ + differential from multiplication. FFA a resolution.

We can define the same transformation for elements of $\mathcal{O}ps$ for well–presented Feynman categories

- The result of a Feynman transform is an *op* over the odd version of the Feynman category
- For the freeness we need model structures, which we give.

Master equations

The Feynman transform is quasi–free. An algebra over $F\mathcal{O}$ is dg–if and only if it satisfies the following master equation.

Name of $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$	Algebraic Structure of $F\mathcal{O}$	Master Equation (ME)
operad ,[GJ94]	odd pre-Lie	$d(-) + - \circ - = 0$
cyclic operad [GK95]	odd Lie	$d(-) + \frac{1}{2}[-,-] = 0$
modular operad [GK98]	odd Lie $+ \Delta$	$d(-) + \frac{1}{2}[-,-] + \Delta(-) = 0$
properad [Val07]	odd pre-Lie	$d(-) + - \circ - = 0$
wheeled properad [MMS09]	odd pre-Lie $+ \Delta$	$d(-) + - \circ - + \Delta(-) = 0$
wheeled prop [KWZ12]	dgBV	$d(-) + \frac{1}{2}[-,-] + \Delta(-) = 0$

Geometry and moduli spaces

Modular Operads

The typical topological example are $\bar{M}_{gn}$. These give rise to chain and homology operads.

- Gromov–Witten invariants make $H^*(V)$ and algebra over $H_*(\bar{M}_{g,n})$

Odd Modular

The canonical geometry is given by $\bar{M}^{KSV}$ which are real blowups of $\bar{M}_{gn}$ along the boundary divisors.

- We get 1-parameter gluings parameterized by S^1. Taking the full S^1 family on chains or homology gives us the structure of an odd modular operad.
- Going back to Sen and Zwiebach, a viable string field theory action S is a solution of the quantum master equation.

Next steps

- Construct Feynman category for the open/closed version of Homological Mirror symmetry.
- Find action of Grothendieck-Teichmüller group (GT).
- Find out the role of fibre functors.
- Connect to GT action (Kitchloo-Morava) on the stable symplectic category.
- ...

The end

Thank you!

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