Value distribution of long Dirichlet polynomials and applications to $\zeta(s)$

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IAS

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Introduction

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- Analytic continuation to all of $\mathbb{C}$ (Riemann).
- Most interesting in the region $\frac{1}{2} \leq \Re s \leq 1$ (the critical strip). The line $\Re s = \frac{1}{2}$ is of particular interest (Riemann Hypothesis).

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Theorem (Bohr-Jessen)

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$$\frac{1}{T} \underset{T \leq t \leq 2T}{meas} \{ \log \zeta(\sigma + it) \in \mathcal{R} \} \to V_\sigma(\mathcal{R})$$

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- The p^{-it} equidistribute when $t \to \infty$.
- Therefore a good model for the average behavior of $\log \zeta(\sigma + it)$ is

$$\sum_{p \leq t} \frac{X(p)}{p^{\sigma}}.$$

with $X(p)$ independent and uniformly distribution on $\mathbb{T}$.

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- For $\sigma = \frac{1}{2}$, the variance is $\sum_{p<t} p^{-1} \sim \log\log t \to \infty$ as $t \to \infty$. Therefore we have a central limit theorem

$$\mathbb{P}\Big(\frac{\sum_{p\le t} \frac{X(p)}{p^{\sigma}}}{\sqrt{\log\log t}} \in \mathcal{R}\Big) \to \iint_{\mathcal{R}} e^{-|z|^2/2} \cdot \frac{|dz|}{2\pi}.$$

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We have,

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$$\frac{T\log T}{\sqrt{\log\log T}} \cdot \exp\big(-\mathcal{L}^2\big) \ll X(T) \ll \frac{T\log T}{\sqrt{\log\log T}} \cdot \mathcal{L}.$$

Old method

- We write

$$\log \zeta(s) = A_X(s) + \mathcal{E}_X(s) \ , \ A_X(s) := \sum_{p \le X} \frac{1}{p^s}$$

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- Pick $X = T^{\varepsilon(T)}$ with $\varepsilon(T) \to 0$ as $T \to \infty$. Thus $|\mathcal{E}_X(s)| \to \infty$ on average. This is where we loose compared to the probabilistic model.

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I'm working on a method that in some cases allows to work with long Dirichlet polynomials

$$\sum_{p \le T^{\delta}} \frac{1}{p^s} \text{ , } \delta > 0 \text{ fixed}$$

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This removes the loss in some of the previous theorems but we need to assume the Riemann Hypothesis. For example,

Theorem (R+,2013)

Assume the Riemann Hypothesis. Let $X(T)$ be the number of sign changes of $S(t) := \frac{1}{\pi}\Im \log \zeta(\frac{1}{2}+it)$ in an interval of length T. Then

$$X(T) \asymp \frac{T \log T}{\sqrt{\log \log T}}.$$

Theorem (R.,2013+)

Assume the Riemann Hypothesis. Let $\mathcal{R}$ be a rectangle with sides $> C$ with C an absolute constant. Then,

$$\underset{T \leq t \leq 2T}{meas} \{ \log \zeta(\tfrac{1}{2} + it) \in \mathcal{R} \} \asymp \frac{T\, meas\{\mathcal{R}\}}{\log\log T}$$

provided that the vertices of the rectangle $\mathcal{R}$ are $o(\sqrt{\log\log T})$.

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- This is a weak form of Ramachandra's conjecture asserting that $\log \zeta(\frac{1}{2} + it)$ (or $\zeta(\frac{1}{2} + it)$) is dense in $\mathbb{C}$.
- The corresponding statement for $\log \zeta(\sigma + it)$ with $\sigma > \frac{1}{2}$ is known since the times of Bohr and Jessen.

Let $s = \frac{1}{2} + it$. The idea is to write,

$$\sum_{p \le T^{\delta}} \frac{1}{p^s} = \sum_i A_i(s) \text{ with } A_i(s) = \sum_{T_{i-1} < p \le T_i} \frac{1}{p^s}$$

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and $T_i = \exp\left(\frac{\log T}{(\log_{i+1} T)^{10}}\right)$. with $\log_{i+1} T$ the $(i+1)$-th iterated logarithm.

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- We look at the joint distribution of the $A_i(s)$'s localizing each $A_i(s)$ to $\alpha_i + O(1/V_i)$.
- This gives rise to a value s at which

$$\sum_{p \le T^\delta} \frac{1}{p^s} = \sum_i \alpha_i + O(1).$$

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- Independently, in a paper of Haper (on arxiv) and R. with Soundararajan (forthcoming).

Thank you!