Quantum Hall Phases, plasma analogy and incompressibility estimates

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References

N. Rougerie, S. Serfaty, J.Y., *Quantum Hall states of bosons in rotating anharmonic traps, Phys. Rev. A* **87**, 023618 (2013), arXiv:1212.1085

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(DOI) 10.1007/s10955-013-0766-0 (Special issue for Herbert Spohn), arXiv:1301.1043

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Fully correlated states in LLL

In the study of the Fractional Hall effect, but also for rapidly rotating Bose gases in harmonic traps, N-particle wave functions on $\mathbb{R}^{2N} \simeq \mathbf{C}^N$ of the following form play an important role:

$$\Psi(z_1, \ldots z_N) = \phi(z_1, \ldots, z_N)\Psi_{\text{Laugh}}(z_1, \ldots z_N)$$

with ϕ analytic and symmetric, and the Laughlin state

$$\Psi^{\ell}_{\text{Laugh}}(z_1, \ldots z_N) = c \prod_{i<j} (z_i - z_j)^{\ell} \, e^{-\sum_{j=1}^{N} |z_j|^2/2}.$$

Here ℓ is odd and ≥ 3 for fermions and ℓ even and ≥ 2 for bosons.

Such functions lie in the lowest Landau level (LLL) of a magnetic Hamilonian and form the kernel $\mathrm{Ker}(\mathcal{I}_N)$ of the contact interaction

$$\mathcal{I}_N = g \sum_{i<j} \delta(z_i - z_j).$$

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Their energy in an external potential V is thus simply given by

$$\mathcal{E}[\Psi] = N \int V(z) \rho_\Psi(z) dz$$

with

$$\rho_\Psi(z) = \int_{\mathbb{C}^{N-1}} |\Psi(z, z_2, \dots, z_N)|^2 \, dz_2 \cdots dz_N.$$

The variational problem to determine

$$E(N,V) = \inf \left\{ \mathcal{E}[\Psi],\ \Psi \in \mathrm{Ker}(\mathcal{I}_N), \|\Psi\|_{L^2(\mathbb{R}^2)} = 1 \right\}$$

and describe the minimizers occurs both in the theory of the FQHE and for rotating Bose gases, where $V(z) = V_{\mathrm{trap}}(z) - \Omega_{\mathrm{rot}}|z|^2$.

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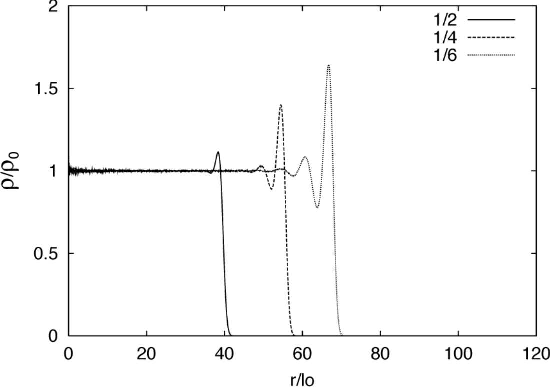
This cannot hold pointwise, however!



1/2
1/4
1/6
2
1.5
1
0.5
0
ρ/ρ0
0
20
40
60
80
100
120
r/lo

We want to derive weak incompressibility bounds for special states and investigate their consequences for the character of the ground state.

An important example, occurring in the theory of fast rotating Bose gases, is

$$V_{\omega,k}(z) = \omega\,|z|^2 + k\,|z|^4.$$

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We would like to compute

$$E_{\omega,k}(N) = \inf\left\{ \int V_{\omega,k}(z)\rho_\Psi(z)\,dz,\ \Psi \in \mathrm{Ker}(\mathcal{I}_N), \|\Psi\|_{L^2(\mathbb{R}^2)} = 1 \right\}.$$

and use this to find criteria for the Hamiltonian, including the interaction, to have a fully correlated ground state in the limit $N \to \infty$.

Hamiltonian in terms of angular momentum.

With $\mathcal{L} = z\partial$ on the Bargmann space $\mathcal{B}$ of analytic functions we have by partial integration

$$\langle \varphi, \mathcal{L}\varphi \rangle = \int |\varphi(z)|^2 (|z|^2 - 1) \exp(-|z|^2) \, \mathrm{d}^2 z$$

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and

$$\langle \varphi, \mathcal{L}^2 \varphi \rangle = \int (|z|^4 - 3|z|^2 + 1) |\varphi(z)|^2 \exp(-|z|^2)\, \mathrm{d}^2 z$$

The Hamiltonian for $k = 0$ is, apart from a constant, can therefore be written

$$H_N = \omega \, \mathcal{L}_N + g \, \mathcal{I}_N$$

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$$\mathcal{L}_N = \sum_{i=1}^{N} z_i \partial_i \qquad \mathcal{I}_N = \sum_{i<j} \delta_{ij}$$

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With $k > 0$ it is

$$H_N = (\omega + 3k)\mathcal{L}_N + k \sum_{i=1}^{N} \mathcal{L}_{(i)}^2 + g\,\mathcal{I}_N$$

Yrast spectrum

Notable feature: The operators $\mathcal{L}_N$ and $\mathcal{I}_N$ commute. The lower boundary of (the convex hull of) their joint spectrum is called the Yrast curve.

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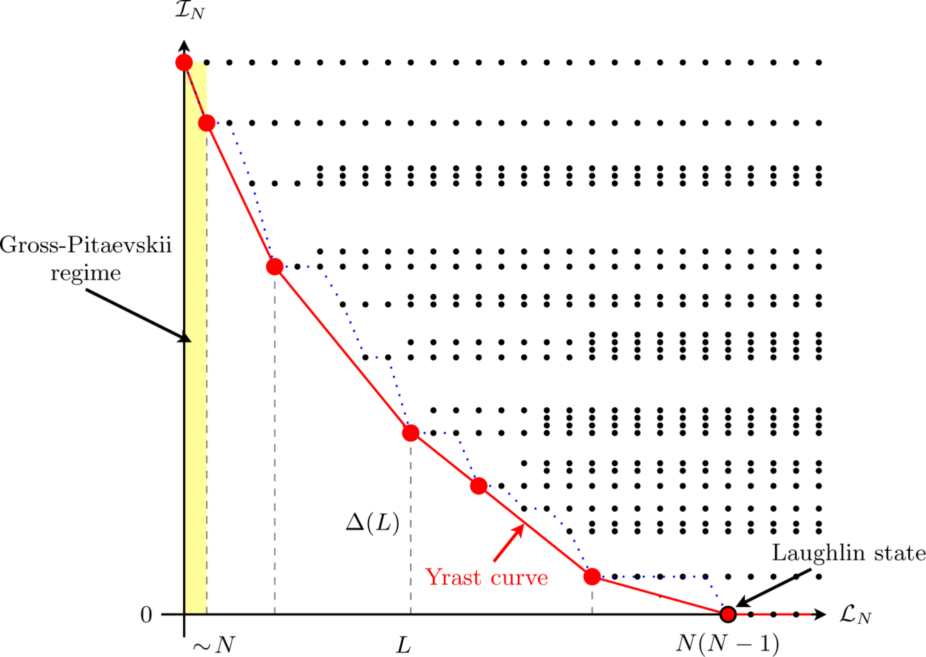
The Laughlin state is an exact ground state of the Hamiltonian with $k = 0$ with angular momentum $N(N-1)$.

On the other hand, $\sum_{i=1}^{N} \mathcal{L}_{(i)}^2$ does not commute with $\mathcal{I}_N$ and the Laughlin state is not an eigenstate of the Hamiltonian with $k > 0$.



$\mathcal{I}_N$
Gross-Pitaevskii
regime
$\Delta(L)$
Yrast curve
Laughlin state
0
$\mathcal{L}_N$
$\sim N$
L
$N(N-1)$

Gaps

For every value of the angular momentum L the interaction operator $\mathcal{I}_N$ has a nonzero spectral gap

$$\Delta_N(L) = \inf\{\text{spec}\, \mathcal{I}_N \restriction_{\mathcal{L}_N = L} \setminus \{0\}\} > 0$$

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The gap, and hence the Yrast curve, are *monotonously decreasing* with L. Reason: the angular momentum of an eigenstate of $\mathcal{I}_N$ can be increased by one unit by multiplying with the center of mass coordinate $(z_1 + \cdots + z_N)/N$.

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There is numerical and some theoretical evidence that

$$\Delta_N(L) \geq \Delta_N(N(N-1) - N) \equiv \Delta > 0$$

for all L independently of N but this is still not proved.
("Gap conjecture".)

Trial States

For an upper bound to the energy for $k > 0$ we consider trial states of the form 'giant vortex times Laughlin', namely, with $m \geq 0$ and $c_{m,N}$ a normalization constant,

$$\Psi_{\mathrm{gv}}^{(m)}(z_1, \ldots, z_N) = c_{m,N} \prod_{j=1}^{N} z_j^m \prod_{i<j} (z_i - z_j)^2 \, e^{-\sum_{j=1}^{N} |z_j|^2/2}$$

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We want to determine the 1-particle density ρ_Ψ of such states and the corresponding energy $\int V_{\omega,k} \, \rho_\Psi$.

The N-particle density as a Gibbs measure

We denote $(z_1, \dots, z_N)$ by Z for short and consider the scaled N particle density (normalized to 1)

$$\mu_{N,m}(Z) := N^N \left| \Psi_{\rm gv}^{(m)}(\sqrt{N} Z) \right|^2 .$$

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We can write

$$\mu_{N,m}(Z) = \mathcal{Z}_{N,m}^{-1} \exp\left(\sum_{j=1}^{N} \left(-N|z_j|^2 + 2m \log |z_j| \right) - 4 \sum_{i<j} \log |z_i - z_j| \right)$$

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with $T = N^{-1}$ and

$$\mathcal{H}_{N,m}(Z) = \sum_{j=1}^{N} \left(-|z_j|^2 + \frac{2m}{N} \log |z_j| \right) - \frac{4}{N} \sum_{i<j} \log |z_i - z_j| .$$

Plasma analogy and mean field limit

The Hamiltonian $\mathcal{H}_{N,m}(Z)$ defines a classical 2D Coulomb gas ('plasma') in a uniform background of opposite charge and a point charge $(2m/N)$ at the origin, corresponding respectively to the $-|z_i|^2$ and the $\frac{2m}{N}\log|z_j|$ terms.

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The probability measure $\mu_{N,m}(Z)$ minimizes the free energy functional

$$\mathcal{F}(\mu) = \int \mathcal{H}_{N,m}(Z)\mu(Z) + T\int \mu(Z)\log\mu(Z)$$

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The $N \to \infty$ limit is in this interpretation a mean field limit where at the same time $T \to 0$. It is thus not unreasonable to expect that for large N, in a suitable sense

$$\mu_{N,m} \approx \rho^{\otimes N}$$

with a one-particle density ρ minimizing a mean field free energy functional.

Mean field limit (cont.)

The mean field free energy functional is defined as

$$\mathcal{E}^{\rm MF}_{N,m}[\rho] := \int_{\mathbb{R}^2} W_m\,\rho + 2\int\int \rho(z)\log|z-z'|\rho(z') + N^{-1}\int_{\mathbb{R}^2}\rho\log\rho$$

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It has a minimizer $\rho^{\rm MF}_{N,m}$ among probability measures on $\mathbb{R}^2$ and this minimizer should be a good approximation for the scaled 1-particle density of the trial wave function, i.e.,

$$\mu^{(1)}_{N,m}(z) := \int_{\mathbb{R}^{2(N-1)}} \mu_{N,m}(z, z_2, \dots, z_N)\mathrm{d}^2 z_2 \dots \mathrm{d}^2 z_N.$$

The Mean Field Limit Theorem

H. Spohn and M. Kiessling have previously studied such mean field limits, using compactness arguments. For our purpose, however, we need quantitative estimates on the approximation of $\mu^{(1)}_{N,m}$ by $\rho^{\mathrm{MF}}_{N,m}$.

THEOREM 2

There exists a constant $C > 0$ such that for large enough N and any $V \in H^1(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$

$$\left| \int_{\mathbb{R}^2} \left(\mu^{(1)}_{N,m} - \rho^{\mathrm{MF}}_{N,m} \right) V \right| \leq C(\log N/N)^{1/2} \|\nabla V\|_{L^1} + CN^{-1} \|\nabla^2 V\|_{L^\infty}$$

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if $m \lesssim N^2$,and

$$\left| \int_{\mathbb{R}^2} \left(\mu^{(1)}_{N,m} - \rho^{\mathrm{MF}}_{N,m} \right) V \right| \leq CN^{-1/2} m^{-1/4} \|V\|_{L_\infty}$$

for $m \gg N^2$.

Ingredients of the Proof of Theorem 2

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The estimate on the density (for $m \lesssim N^2$) follows essentially from the fact that the positive Coulomb energy $D(\mu^{(1)} - \rho^{\mathrm{MF}}, \mu^{(1)} - \rho^{\mathrm{MF}})$ is squeezed between the upper and lower bounds to the free energy.

Properties of the Mean Field Density

The picture of the 1-particle density arises from asymptotic formulas for the mean-field density: If $m \leq N^2$, then ρ_m^{MF} is well approximated by a density $\hat{\rho}_m^{\mathrm{MF}}$ that minimizes the mean field functional without the entropy term.

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Variational equation satisfied by this density:

$$|z|^2 - 2(m/N)\log|z| + 4\rho * \log|z| - C \geq 0$$

mit "$=$" where $\rho > 0$ and "$>$" where $\rho = 0$.

Taking the Laplacian of the variational equation and using the subharmonicity of $\log|z|$ it follows that the MF density takes a constant value $(2\pi)^{-1}$ on an annulus with inner and outer radia $R_- = (m/N)^{1/2}$ and $R_+ = (2 + m/N)^{1/2}$ and is zero otherwise.

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The density is well approximated by a gaussian $\rho^{\text{th}}(z) \sim |z|^{2m}\exp(-N|z|^2)$ that is centered around $\sqrt{m/N}$.

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$$m_{\mathrm{opt}} = \begin{cases} 0 & \text{if } \omega \geq -2kN \\ \frac{|\omega|}{2k} - N & \text{if } \omega < -2kN. \end{cases}$$

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This is consistent with the picture that the Laughlin state is an approximate ground state also for negative ω as long as $|\omega|/k \lesssim N$. The angular momentum remains $O(N^2)$ in these cases.

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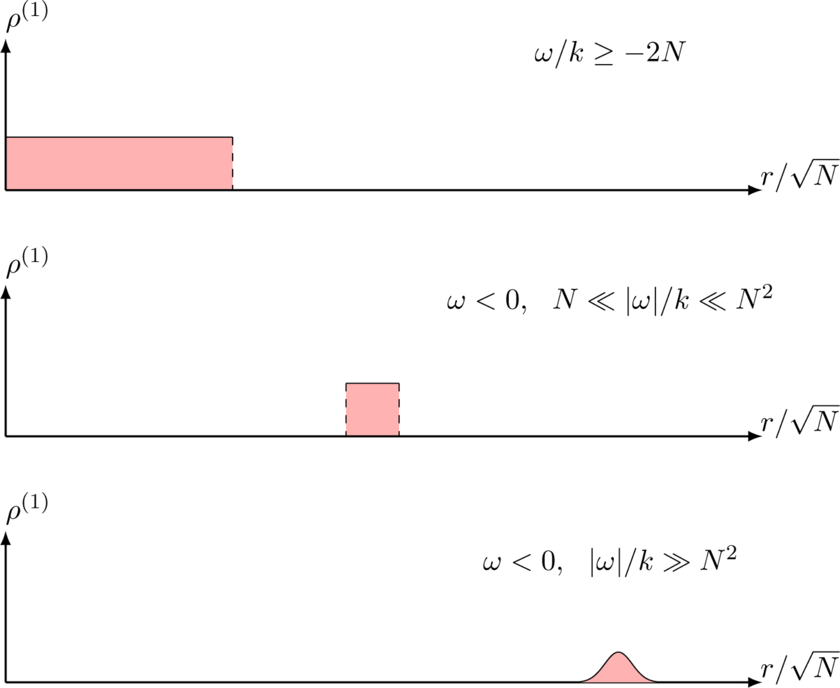
As the parameters ω and k tend to zero and N is large the qualitative properties of the optimal trial wave functions thus exhibit different phases:

- The state changes from a pure Laughlin state to a modified Laughlin state with a ‘hole’ in the density around the center when ω is negative and $|\omega|$ exceeds $2kN$.
- A further transition is indicated at $|\omega| \sim kN^2$. The density profile changes from being ‘flat’ to a Gaussian.


$\rho^{(1)}$
$\omega/k \geq -2N$
$r/\sqrt{N}$
$\rho^{(1)}$
$\omega < 0, \quad N \ll |\omega|/k \ll N^2$
$r/\sqrt{N}$
$\rho^{(1)}$
$\omega < 0, \quad |\omega|/k \gg N^2$
$r/\sqrt{N}$

The upper bounds for the energy obtained from the trial wave functions can be combined with and a simple lower bound that follows from

$$\sum_i \mathcal{L}_{(i)}^2 \geq \frac{1}{N}\left(\sum_i \mathcal{L}_{(i)}\right)^2.$$

This inequality holds because $\mathcal{L}_{(i)}$ and $\mathcal{L}_{(j)}$ commute for any i, j.

The upper bounds for the energy obtained from the trial wave functions can be combined with and a simple lower bound that follows from

$$\sum_i \mathcal{L}_{(i)}^2 \geq \frac{1}{N}\left(\sum_i \mathcal{L}_{(i)}\right)^2 .$$

This inequality holds because $\mathcal{L}_{(i)}$ and $\mathcal{L}_{(j)}$ commute for any i, j.

It leads to lower bounds of the same order of magnitude as the upper bounds and to a sufficient criterion for strong correlations in the ground state:

Criterion for strong correlations in the ground state Ψ_o

THEOREM 1.

$$\left\| P_{\mathrm{Ker}(\mathcal{I}_N)^\perp} \Psi_0 \right\| \to 0$$

in the limit $N \to \infty$, $\omega, k \to 0$ if one of the following conditions hold:

- $\omega \geq 0$ and $\omega N^2 + kN^3 \ll g\,\Delta$.

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Note: For $k = 0$ the first item is just the condition for the passage to the Laughlin state, while the other conditions are void because $\omega < 0$ is only allowed if $k > 0$.

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- The density of the trial wave functions can be analyzed through the plasma analogy. The character of the density changes at $|\omega|/k = O(N)$ and again at $|\omega|/k = O(N^2)$.
- Sharp lower bounds, that would follow from a general proof of incompressibility of fully correlated states, would establish the observed crossing of optimal trial functions as a genuine quantum phase transition.

Incompressibility estimates in the Laughlin phase

Consider now again general states of the form

$$\Psi(z_1, \ldots, z_N) = \phi(z_1, \ldots, z_N) \prod_{i<j} (z_i - z_j)^\ell e^{-\sum_{j=1}^N |z_j|^2/2}$$

with ℓ even ≥ 2 for bosons, or ℓ odd ≥ 3 for fermions, and $\phi(z_1, \ldots, z_N)$ a symmetric polynomial.

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and that the density $\mu^{(1)}$ satisfies the “incompressibility bound”

$$\mu^{(1)}(z)\leq\frac{1}{\ell\pi}.$$

An “incompressibility bound” leads to a lower bound to potential energies

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But:

- The bound will certainly not hold pointwise for finite N.
- A bound for arbitrary ‘correlation factors’ ϕ is out of reach at present.

It is, however, possible to derive such bounds for ϕ of the form

$$\prod_{j=1}^{N} f_1(z_j) \prod_{(i,j)\in\{1,\dots,N\}} f_2(z_i, z_j) \dots \prod_{(i_1,\dots,i_n)\in\{1,\dots,N\}} f_n(z_{i_1}, \dots, z_{i_n})$$

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We have done this explicitly for the special case

$$\prod_{j=1}^{N} f_1(z_j) \prod_{(i,j)\in\{1,\dots,N\}} f_2(z_i, z_j),$$

but extensions to arbitrary fixed n (or n not growing too fast with N) are possible.

For such factors we have proved that the potential energy is, indeed, bounded below by the bathtub energy in the limit $N \to \infty$.

The proof proceeds in two steps:

- Comparison of the free energy with a mean field energy defined by the functional

$$\mathcal{E}^{\mathrm{MF}}[\rho] := \int_{\mathbb{R}^2} \left(|z|^2 - 2\log|g_1(z)|\right)\rho(z)dz + \int_{\mathbb{R}^4} \rho(z)\Big(-\ell\log|z-z'| - \log|g_2(z,z')|\Big)\rho(z')dzdz'$$

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The first step is the tricky part. Note that $-\log|g_2(z,z')|$ is much less simple than $-\log|z-z'|$ and not of positive type in general so standard arguments for the mean field limit do not apply.

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Instead we use a theorem of Diaconis and Freedman:

Theorem (Diaconis-Freedman)

Let S be a measurable space and $\mu \in \mathcal{P}_s(S^N)$ be a probability measure on S^N invariant under permutation of its arguments. There exists $P_\mu \in \mathcal{P}(\mathcal{P}(S))$ a probability measure such that, denoting

$$\tilde{\mu} := \int_{\rho \in \mathcal{P}(S)} \rho^{\otimes n} dP_\mu(\rho)$$

we have

$$\left\| \mu^{(n)} - \tilde{\mu}^{(n)} \right\|_{\mathrm{TV}} \leq \frac{n(n-1)}{N}.$$

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In addition, the marginals of $\tilde{\mu}$ are given by :

$$\tilde{\mu}^{(n)}(x_1, \ldots, x_n) = \frac{1}{N^n} \sum_{j=1}^{n} \sum_{1 \leq i_1 \neq \ldots \neq i_j \leq N} \mu^{(j)}(x_{i_1}, \ldots, x_{i_j})\, \delta_{x_{i_{j+1}} = \ldots = x_{i_n}}.$$

Incompressibility Bound

With the aid of the DF theorem we prove:

Theorem (Incompressibility Bound)

Let Ψ be a wave function in the Laughlin phase with a correlation factor of the type described and $\mu^{(1)}$ its scaled 1-particle probability density. Let V be a smooth potential with $\inf_{|x|\geq R} V(x) \to \infty$ for $R \to \infty$. Then

$$\liminf_{N\to\infty} \int \mu^{(1)}(z)V(z)dz \geq E^{\rm bt}(V).$$

Moreover, for a radially monotone, or Mexican hat shaped potential (like $V_{\omega,k}$) this bound is optimal.

An Open Problem

Show that for general V the the energy in $\mathrm{Ker}(\mathcal{I}_N)$ is, in the limit $N \to \infty$, minimized by wave functions of the form

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In words, the Conjecture is that additional correlations beyond those present in the Laughlin state cannot lower the energy.