

Analysis and Math Physics Seminar, IAS

A variational approach to the regularity theory for the Monge-Ampère equation

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joint work with Michael Goldman (arXiv '17, Ann. ENS),
with MG & Martin Huesmann (arXiv '18, '19)
with Tatsuya Miura (arXiv '20)

Follow de Giorgi's approach for Minimal Surfaces

Harmonic approximation $\xRightarrow{\text{elliptic regularity, change of variables}}$
One-Step improvement $\xRightarrow{\text{Campanato iteration}}$ ϵ -regularity
 $\xRightarrow{\text{measure theory}}$ partial regularity.

Here $C^{2,\alpha}$ -regularity for Kantorowicz potential ψ
i. e. $C^{1,\alpha}$ -regularity for Brenier map $T = \nabla\psi$.

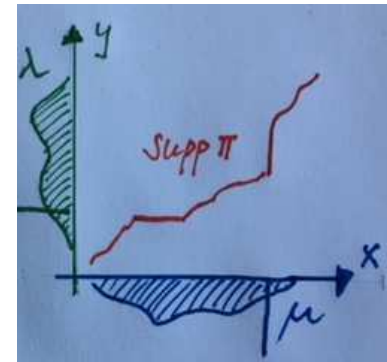
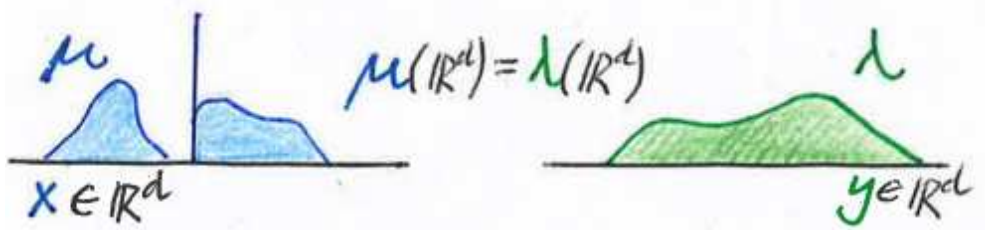
Purely variational/ no maximum principle/ avoids Caffarelli's theory

Applications (old and new):

Partial regularity [Figalli-Kim, DePhilippis-Figalli],
 ϵ -regularity at boundary for $\partial\Omega \in C^{1,\alpha}$ [Chen-Figalli],
“Matching” Poisson to Lebesgue [Parisi et. al, Ambrosio et. al]

Recall Optimal Transportation

Given two measures



seek transfer plan π , i. e. $\pi(U \times \mathbb{R}^d) = \mu(U)$, $\pi(\mathbb{R}^d \times V) = \lambda(V)$
that minimizes Euclidean transport cost $\int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 \pi(dx dy)$.

Minimum $\mathrel{=}\, W^2(\mu, \lambda)$ (squared) Wasserstein distance.

From optimal transportation to Monge-Ampère

Minimize $\int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 \pi(dx dy)$
among all $\pi(dx dy)$ with marginals $\mu(dx)$ and dy .

Support of optimal π is cyclically monotone; hence
 $\exists$ convex ψ $\text{supp} \pi \subset \{ (x, y) \mid y \in \text{subgradient } \partial\psi(x) \}$.

$\forall$ test functions ζ $\int \zeta(\nabla\psi(x)) \mu(dx) = \int \zeta(y) dy$.

In smooth case, this amounts to $\det D^2\psi = \mu$,
an instance of the Monge-Ampère equation.

Nature of the Monge-Ampère equation

Recall Monge-Ampère: $\det D^2\psi = 1$.

Fully non-linear with $F(A) := \det A - 1$.

However elliptic: $F(A) > F(A')$ for $A > A' \geq 0$;
satisfies comparison principle.

However degenerate: $\leftrightarrow$ affine invariant (non-compact $SL(n)$).

Cf. Laplacian $F(A) = \operatorname{tr} A$: rotation invariant (compact $SO(n-1)$).

Caffarelli's '90 breakthrough:

comparison principle, affine invariance, compactness.

Monge-Ampère equation at crossroads
of fully nonlinear and variational.

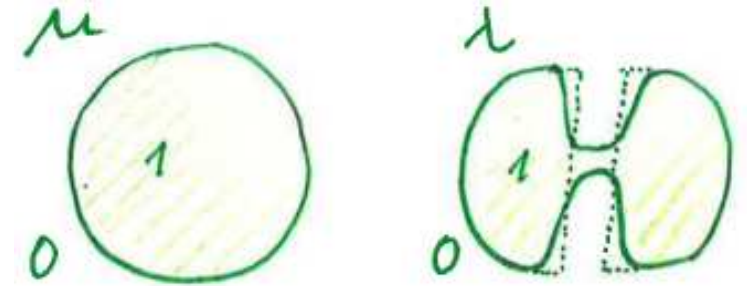
Singularities in optimal transportation are generic

Caffarelli's example:

smooth data μ, λ

do not yield smooth $T = \nabla\psi$.

See also Loeper's example in Riemannian setting.



Thus ϵ -regularity is of interest:

$$\int |y-x|^2 \pi(dxdy) \leq \epsilon \text{ locally, } \mu, \lambda \text{ smooth locally} \\ \implies \text{Kantorowicz potential } \psi \text{ smooth locally.}$$

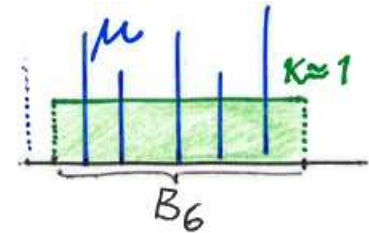
Relies on harmonic approximation in our approach:

$$\int |y-x|^2 \pi(dxdy) \leq \epsilon \text{ locally, } \mu, \lambda \approx 1 \text{ locally} \\ \implies \text{displacement } (y-x) \pi(dxdy) \approx \nabla \text{harmonic locally.}$$

At the core: harmonic approximation

Local energy $E := \int_{(B_6 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_6)} |y - x|^2 \pi(dx dy),$

Local data size² $D := W_{B_6}^2(\mu, \kappa) + (\kappa - 1)^2$
 $+ \text{ same for } \lambda$



Proposition 1 (Goldman&Huesmann&O.)

$$\forall \tau > 0 \quad \exists \epsilon(\tau, d) > 0, C(\tau, d) < \infty \quad \text{s. t.} \quad E + D \leq \epsilon \implies$$

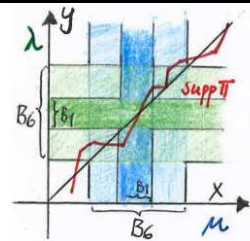
$$\exists \nabla \phi \text{ harmonic,} \quad \int_{B_3} |\nabla \phi|^2 \leq C(E + D),$$

$$\int_{(B_1 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_1)} \left| y - x - \nabla \phi\left(\frac{1}{2}x + \frac{1}{2}y\right) \right|^2 \pi(dx dy) \leq \tau E + CD.$$

Amounts to:

Displacement $y - x$

$\approx$ harmonic gradient $\nabla \phi$



Harmonic approximation: correct homogeneities ...

$E := \int_{(B_6 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_6)} |y - x|^2 \pi(dx dy)$, quadratic in solution,

$D := W_{B_6}^2(\mu, \kappa) + (\kappa - 1)^2 + \text{same for } \lambda$, quadratic in data.

Proposition 1

$\forall \tau > 0 \quad \exists \epsilon(\tau, d) > 0, C(\tau, d) < \infty \quad \text{s. t.} \quad E + D \leq \epsilon \implies$

$\exists \nabla \phi$ harmonic, $\int_{B_2} |\nabla \phi|^2 \leq C(E + D),$

$\int_{(B_1 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_1)} |y - x - \nabla \phi(\frac{1}{2}x + \frac{1}{2}y)|^2 \pi(dx dy) \leq \tau E + CD.$

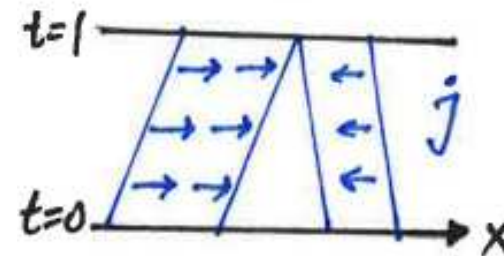
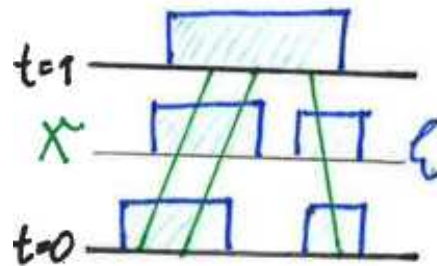
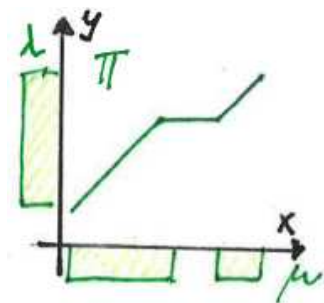
Compare to $\int_{B_1} L(\nabla u - \nabla \phi) \leq \tau \int_{B_6} L(\nabla u) + C \int_{B_6} |f|^2$

for $-\nabla \cdot DL(\nabla u) = \nabla \cdot f$ with uniformly convex L .

... and correct metric

From Lagrangian to Eulerian (Benamou-Brenier)

Transport plan π , trajectories $X(t)$, density/flux (ρ, j)



Continuity eqn. $\partial_t \rho + \nabla \cdot j = 0$, kinetic energy $\int_{\mathbb{R}^d \times (0,1)} \frac{1}{\rho} |j|^2$.

$$W^2(\mu, \lambda) = \inf \left\{ \int \frac{1}{\rho} |j|^2 \mid \partial_t \rho + \nabla \cdot j = 0, \rho(t=0) = \mu, \rho(t=1) = \lambda \right\}$$

Kinetic energy density $(\rho, j) \mapsto \frac{1}{\rho} |j|^2$ is strictly convex.

Linearization $\frac{1}{\rho} |j|^2 \rightsquigarrow |j|^2$ amounts to $W^2(\mu, \lambda) \rightsquigarrow \|\mu - \lambda\|_{\dot{H}^{-1}}^2$.

Analogy for minimal surfaces: varifolds vs. currents.

Eulerian version of harmonic approximation

Density/flux $(\rho, j) = (\rho_t dt, j_t dt)$ where

$$\int \zeta d\rho_t = \int \zeta(ty + (1-t)x) \pi(dxdy), \quad \int \xi \cdot dj_t = \int \xi(ty + (1-t)x) \cdot (y-x) \pi(dxdy)$$

continuity
equation

$$E \rightsquigarrow \int_{B_5 \times (0,1)} \frac{1}{\rho} |j|^2$$

reveals strict convexity
of variational problem

Proposition 1'

$$\forall \tau > 0 \quad \exists \epsilon(\tau, d) > 0, \quad C(\tau, d) < \infty \quad \text{s. t.} \quad E + D \leq \epsilon \implies$$

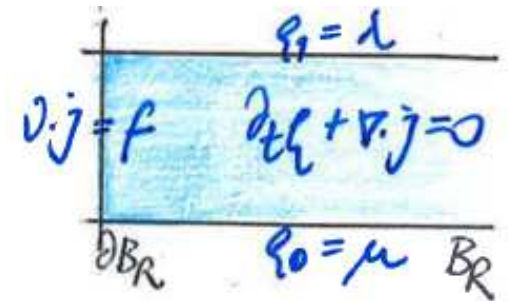
$$\exists \nabla \phi \text{ harmonic,} \quad \int_{B_2} |\nabla \phi|^2 \leq C(E + D),$$

$$\int_{B_2 \times (0,1)} \frac{1}{\rho} |j - \rho \nabla \phi|^2 \leq \tau E + CD.$$

Amounts to: Eulerian velocity $\frac{j}{\rho} \approx \nabla \phi$ harmonic gradient

Construct $\nabla\phi$ via flux (Neumann) data

Normal flux $f := \nu \cdot j$ across bdry ∂B_R ,
its time integral $\bar{f} := \int_0^1 f dt$.



Proposition 1'' $\forall \tau > 0 \exists \epsilon > 0, C < \infty : E + D \leq \epsilon \implies$

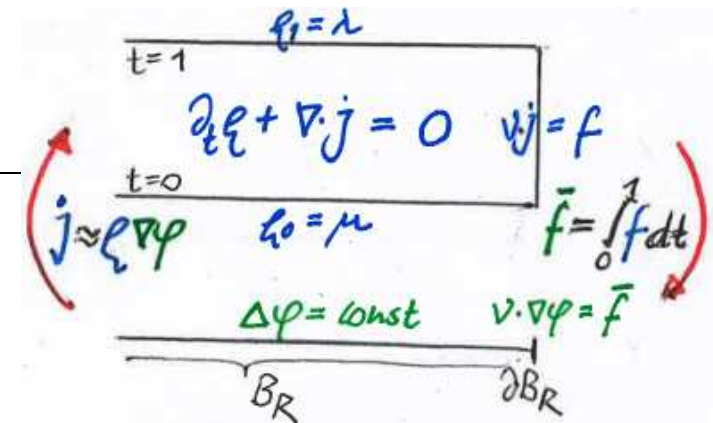
$\exists R \in (3, 4)$ s. t. $\Delta\phi = \text{const}$ in B_R , $\nu \cdot \nabla\phi = \bar{f}$ on ∂B_R

satisfies $\int_{B_2} |\nabla\phi|^2 \leq C(E + D),$

$$\int_{B_2 \times (0,1)} \frac{1}{\rho} |j - \rho \nabla\phi|^2 \leq \tau E + CD.$$

cf. Dacorogna-Moser.

Choice of “good” radius R .



Old application: ϵ -regularity as Schauder theory

Recall Hölder semi-norms $[f]_{\alpha,B} := \sup_{x \neq x' \in B} \frac{|f(x) - f(x')|}{|x - x'|^\alpha}$, $\alpha \in (0, 1)$.

Suppose $\mu = f dx$, $\lambda = g dy$

with Hölder continuous f, g and $f(0) = g(0) = 1$.

Monitor (the dimensionless) $E := \frac{1}{R^2 |B_R|} \int_{\Omega \cap B_{2R}} |T - x|^2 dx$
with $T = \nabla \psi$ Brenier map.

Monitor (the dimensionless) $D := R^{2\alpha} [f]_{\alpha, B_{2R}}^2 + R^{2\alpha} [g]_{\alpha, B_{2R}}^2$.

Theorem 1 (Goldman&O., à la DePhilippis&Figalli)

If $E + D \ll 1$ then $R^{2\alpha} [\nabla T]_{\alpha, B_R}^2 \lesssim E + D$.

Amounts to $C^{2,\alpha}$ -regularity for Monge-Ampère $\det D^2 \psi = f$ ($g \equiv 1$).

Comparison: DePhilippis&Figalli vs. Goldman&O.

Theorem 1 (Goldman&O., à la DePhilippis&Figalli)

$$E + [f]_{\alpha, B_2}^2 + [g]_{\alpha, B_2}^2 \ll 1 \implies [D^2\psi]_{\alpha, B_1}^2 \lesssim E + [f]_{\alpha, B_2}^2 + [g]_{\alpha, B_2}^2.$$

Perturbation around **linear** $\Delta\phi = \text{const}$,

as opposed to **nonlinear** $\det D^2\psi = 1$ (ϵ -regularity Figalli&Kim).

Get (immediately) linear homogeneities $[D^2\psi]_{\alpha, B_1} \lesssim [f]_{\alpha, B_2} + [g]_{\alpha, B_2}$.

Get $\psi \in C^{2,\alpha}$ **in one step**,

as opposed to **three-step bootstrap** $C^{1,\alpha}, C^{1,1}, C^{2,\alpha}$.

Competitors in strictly convex variational problem,
instead of **comparison principle**.

New application: Boundary ϵ -regularity

Suppose $\mu = dx|_{\Omega}$, $\lambda = dy|_{\Lambda}$ for two open sets $\Omega, \Lambda \subset \mathbb{R}^d$.

Consider $E := \frac{1}{R^2|B_R|} \int_{\Omega \cap B_{2R}} |T - x|^2 dx$ where T Brenier map

Topological condition:

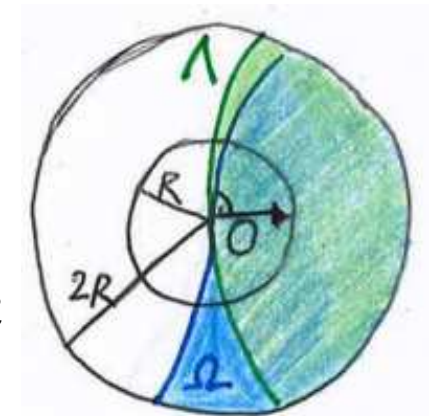
$$T(\Omega \cap B_R) \subset \Lambda \cap B_{2R}, \quad T^{-1}(\Lambda \cap B_R) \subset \Omega \cap B_{2R}.$$

Geometric condition:

$$\Omega \cap B_{2R} = \{x_1 > g(x')\} \cap B_{2R}, \quad \Lambda \cap B_{2R} = \{x_1 > h(x')\} \cap B_{2R}$$

with $g(0) = h(0) = 0$, $\nabla' g(0) = \nabla' h(0) = 0$.

$$\text{Consider } D := R^{2\alpha} [\nabla' g]_{\alpha, B_{2R}}^2 + R^{2\alpha} [\nabla' h]_{\alpha, B_{2R}}^2.$$



Theorem 2 (Miura&O., à la Chen-Figalli)

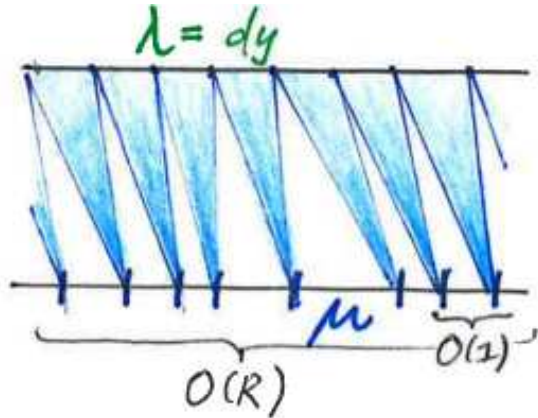
$$\text{If } E + D \ll 1 \quad \text{then} \quad R^{2\alpha} [\nabla T]_{\alpha, B_R}^2 \lesssim E + D$$

Optimal by Jhaveri '19

New application: Quantitative linearization ...

Now general μ but $\lambda = dy$ Lebesgue. Expect:

$$\mu \approx dx \implies y - x \approx \nabla u \text{ under } \pi, \text{ where } \Delta u = \mu - 1.$$



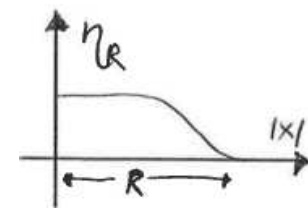
With an eye on
matching Poisson (scale 1)
to Lebesgue (on scale $R \gg 1$),

want: $\mu \approx dx$ on large-scale average,

$$\text{eg. control of } \frac{1}{|B_R|} W_{B_R}^2(\mu, \kappa) + (\kappa - 1)^2$$

$\implies y - x \approx \nabla u$ on large-scale average,

$$\text{eg. control of } \int \eta_R (y - x) \pi(dx dy) - \int \eta_R \nabla u.$$



... on large scales

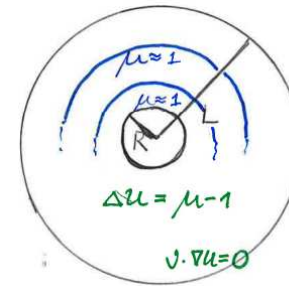
Quantitative linearization on large scales

Admissible rate function:

$$\frac{1}{C} \sum_{k=0}^{\infty} \frac{\beta(2^k R)}{2^k R} \leq \frac{\beta(R)}{R} \leq C \quad \text{for } R \geq 1.$$

Poisson: $\beta(R) \sim \ln R$ for $d = 2$,

$\beta(R) \sim 1$ for $d > 2$.



Theorem 3 (Goldman&Huesmann&O.)

Suppose $\text{supp} \mu \subset B_L$, $\mu(\mathbb{R}^d) = |B_L|$

and $\frac{1}{|B_R|} W_{B_R}^2(\mu, \kappa) + (\kappa - 1)^2 \leq \beta(R)$ for all $1 \leq R \leq L$.

Then $\left| \frac{\int \eta_R(x) (y - x) \pi(dx dy)}{\int \eta_R(x) \pi(dx dy)} - \frac{\int \eta_R \nabla u}{\int \eta_R} \right| \lesssim \frac{\beta(R)}{R}$, $1 \ll R \leq L$,

where $\Delta u = \mu - 1$ in B_L , $\nu \cdot \nabla u = 0$ on ∂B_L .

Harmonic approximation even interesting for μ, λ (locally) constant

Suppose $\mu = dx$ and $\lambda = dy$ on B_6 i.e. $D = 0$

Recall: $f := \nu \cdot j$, $\bar{f} := \int_0^1 f dt$, $E := \int_{B_5 \times (0,1)} \frac{1}{\rho} |j|^2$.

Lemma 1 (Goldman&O.)

$\exists R \in (3, 4)$ s. t. $\left\{ \begin{array}{l} \Delta \phi = 0 \text{ in } B_R, \\ \nu \cdot \nabla \phi = \bar{f} \text{ on } \partial B_R \end{array} \right\}$ satisfies

$$\int_{B_R} |\nabla \phi|^2 \leq C_{(d)} E,$$

$$\int_{B_R \times (0,1)} \frac{1}{\rho} |j - \rho \nabla \phi|^2 \leq C_{(d)} E^{\frac{d+2}{d+1}}.$$

transport velocity $\frac{j}{\rho} \approx \nabla \phi$ gradient of harmonic function

Reflects that Wasserstein $W_2 \approx \dot{H}^{-1}$ for densities ≈ 1 .

Ingredients for Lemma 1

Lemma 1: $\exists \phi$ harmonic such that

$$\int_0^1 \int_{B_1} \frac{1}{\rho} |j - \rho \nabla \phi|^2 \lesssim \left(\int_0^1 \int_{B_2} \frac{1}{\rho} |j|^2 \right)^{\frac{d+2}{d+1}}.$$

1) Construction of ϕ : $\Delta \phi = 0$ in B_R , $\nu \cdot \nabla \phi = \bar{f}$ on ∂B_R ,
where $\bar{f} := \int_0^1 f$, $f := \nu \cdot j$

and $R \in [1, 2]$ such that $\int_0^1 \int_{\partial B_R} \frac{1}{\rho} |j|^2 \lesssim \int_0^1 \int_{B_2} \frac{1}{\rho} |j|^2$ (good radius)

2) Orthogonality:

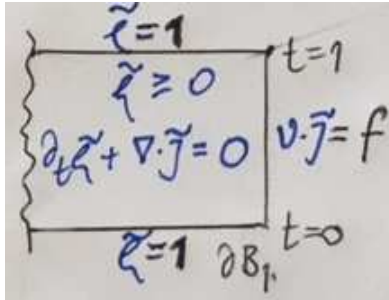
$$\int_0^1 \int_{B_R} \frac{1}{\rho} |j - \rho \nabla \phi|^2 \leq \int_0^1 \int_{B_R} \frac{1}{\rho} |j|^2 - \int_{B_R} |\nabla \phi|^2.$$

Follows from McCann's *displacement convexity* in form of $\rho \leq 1$.

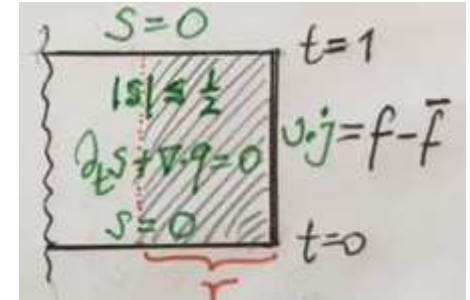
3) Construction of competitor: $(\tilde{\rho}, \tilde{j})$ admissible with bc

of (ρ, j) and $\int_0^1 \int_{B_R} \frac{1}{\tilde{\rho}} |\tilde{j}|^2 - \int_{B_R} |\nabla \phi|^2 \lesssim \left(\int_0^1 \int_{\partial B_R} f^2 \right)^{\frac{d+2}{d+1}}.$

Ingredients for Lemma 1: boundary layer construction



Ansatz $(\tilde{\rho}, \tilde{j}) = (1, \nabla \phi) + (s, q)$
 $\text{supp}(s, q) \subset [0, 1] \times (\bar{B}_R - B_{R-r})$



Enough to construct (s, q) such that $|s| \leq \frac{1}{2}$ and

$$\int_0^1 \int_{B_R} |q|^2 \lesssim r \int_0^1 \int_{\partial B_R} (f - \bar{f})^2 \text{ for } r \gg \left(\int_0^1 \int_{\partial B_R} (f - \bar{f})^2 \right)^{\frac{1}{d+1}}.$$

Reduce to *isoperimetric estimate* $\left(\int_0^1 \int_{\partial B_R} (\zeta - \bar{\zeta})^2 \right)^{\frac{1}{2}}$

$$\lesssim r^{\frac{1}{2}} \left(\int_0^1 \int_{B_R - B_{R-r}} |\nabla \zeta|^2 \right)^{\frac{1}{2}} + r^{-\frac{d+1}{2}} \int_0^1 \int_{B_R - B_{R-r}} |\partial_t \zeta|.$$

Reminiscent of Alberti & Choksi & 0. '09.

Analogies to minimal surfaces (Schoen&Simon '82)

Approximate minimal surface by harmonic graph /
approximate displacement by harmonic gradient.

Use: Object is minimizing under compact perturbations.
Don't use: Euler-Lagrange equation (= first variation).

Mismatch of type of boundary condition
for construction of harmonic competitor:
graph vs. non-graph / time-averaged vs. time-resolved;

Lower-dimensional isoperimetric estimate:
error is of higher-order (choice of good radius).

Use of strict convexity to convert energy gap
into distance (“approximate orthogonality”);
need to smooth out boundary data.

From Lemma 1 to Theorem 1

Inner regularity for ϕ . Eulerian $(\rho, j) \rightsquigarrow$ Lagrangian T .

For $Q \in \mathbb{R}^{d \times d}$, $b \in \mathbb{R}^d$ consider

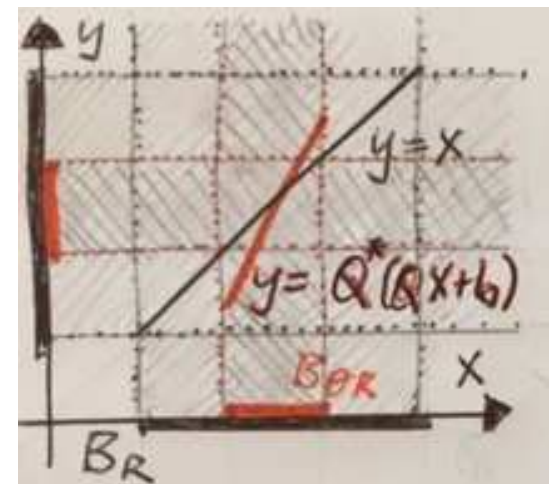
$$E(R, Q, b) := \frac{1}{R^{d+2}} \int_{B_R} |Q^{-*}T - (Qx + b)|^2 dx$$

$$E(R) := E(R, \text{id}, 0) \ll 1 \implies$$

$$\exists Q, b \quad \forall 0 < \theta \ll 1$$

$$E(\theta R, Q, b) \lesssim \theta^2 E(R) + \theta^{-(1+\frac{d}{2})} E^{\frac{d+2}{d+1}}(R)$$

$$|Q - \text{id}|^2 + \frac{1}{R^2} |b|^2 \lesssim E(R)$$



Affine change of variables $\hat{x} = Qx + b$, $\hat{y} = Q^{-*}y$

preserves optimality of transport. Campanato iteration.

Changes for rough μ , λ , flux boundary data

Recall: $D := W_{B_6}^2(\mu, \kappa) + (\kappa - 1)^2$, $E := \int_{(B_6 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_6)} |y - x|^2 \pi(dx dy)$.

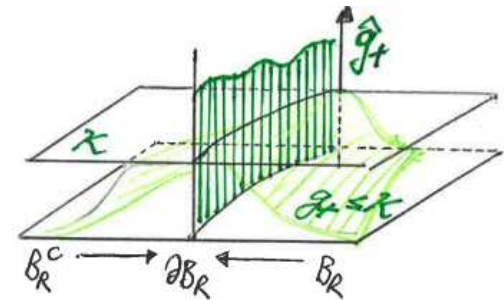
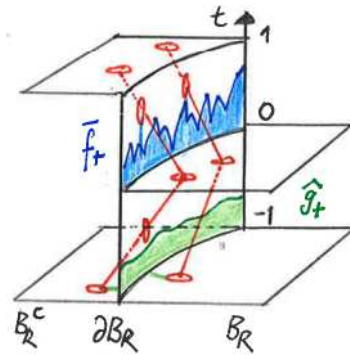
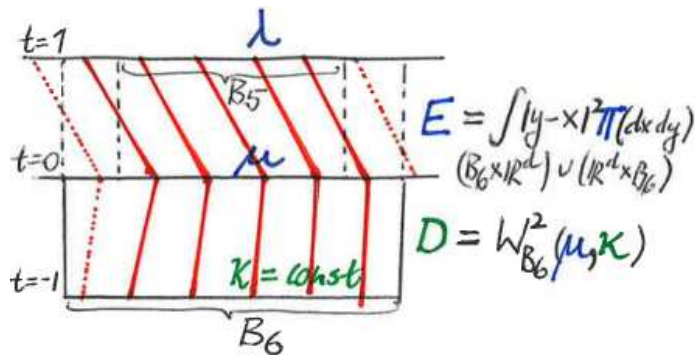
Issue: $\bar{f} = \int_0^1 f dt$, $f = \nu \cdot j$ **measure** $\notin L^2$.

Replace $\bar{f}$ by $\hat{g} \in L^2$ in

$$\Delta \phi = c \text{ in } B_R, \quad \nu \cdot \nabla \phi = \hat{g} = \hat{g}_+ - \hat{g}_- \text{ on } \partial B_R$$

Lemma (nonlin. L^2 -approx.) $\exists R \in (3, 4)$, $\hat{g}_{\pm}$ on ∂B_R s. t.

$$\int_{\partial B_R} \hat{g}_{\pm}^2 \lesssim E + D, \quad W_{\partial B_R}^2(\hat{g}_{\pm}, \bar{f}_{\pm}) \lesssim (E + D)^{1 + \frac{1}{d+2}}$$



Changes for rough μ, λ , L^∞ -bound

Recall: $D := W_{B_6}^2(\mu, \kappa) + (\kappa - 1)^2$, $E := \int_{(B_6 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_6)} |y - x|^2 \pi(dx dy)$.

Lemma (nonlinear L^2 -approximation)

$\exists R \in (3, 4)$, $\hat{g}_\pm$ on ∂B_R s. t.

$$\int_{\partial B_R} \hat{g}_\pm^2 \lesssim E + D, \quad W_{\partial B_R}^2(\hat{g}_\pm, \bar{f}_\pm) \lesssim (E + D)^{1 + \frac{1}{d+2}}$$

Superlinear **exponent** comes from

Lemma (L^∞ -estimate)

For $(x_0, x_1) \in \text{supp } \pi \cap (B_5 \times \mathbb{R}^d) \cup (\mathbb{R}^d \times B_5)$

$$|x_0 - x_1| \lesssim (E + D)^{\frac{1}{d+2}}$$

Follows from monotonicity of $\text{supp } \pi$

Summary and outlook

Harmonic approximation $\implies$ One-Step improvement
Campanato iteration $\implies$ small or large-scale-regularity

Applications: ϵ -regularity at boundary for $\partial\Omega \in C^{1,\alpha}$

[Caffarelli, Savin-Yu, Chen-Figalli, Chen-Liu-Wang],

Matching Poisson to Lebesgue

[Sturm-Huesmann, Parisi et.al, Ambrosio et.al, Ledoux, Talagrand]

Next steps:

general cost functions with Prodhomme & Ried,

thermodynamic limit in Matching in $d = 2$.