Calabi–Yau mirror symmetry from categories to curve-counts

Tim Perutz (University of Texas at Austin)
Joint work with Nick Sheridan (Princeton/IAS)

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A and B

- Mirror symmetry relates the *A-model* geometry of one (suitably enhanced) Calabi–Yau manifold X to the *B-model* geometry of another, $\check{X}$.
- (A) symplectic
  Fukaya categories ("open string" invariants)
  Quantum cohomology,
  counting holomorphic curves ("closed string" invariants)
  (B) algebro-geometric
  derived categories of coh. sheaves ("open string invariants")
  the Gauss–Manin connection in de Rham cohomology
  periods of a volume-form; ("closed string invariants")
- Mirror symmetry is involutory: $\check{\check{X}} = X$. In my presentation, A and B will appear asymmetric, because I'm only showing you part of the data.

Gist of the lecture

- Mirror CY pairs
  X: symplectic manifold with $c_1(TX) = 0$;
  $\check{\mathcal{X}}$: smooth projective variety with trivialized canonical bundle.
- Open string (homological) mirror symmetry claims:

  derived Fukaya category $\mathcal{F}(X)$ is equivalent to

  derived category of coherent sheaves $D^b \operatorname{Coh}(\check{\mathcal{X}})$

- We'll assume

  subcat. $\mathcal{A}$ of $\mathcal{F}(X) \simeq$ subcat. $\mathcal{B}$ of $D^b \operatorname{Coh}(\check{\mathcal{X}})$

  where $\mathcal{B}$ generates $D^b \operatorname{Coh}(\check{\mathcal{X}})$.

- From this we'll deduce homological mirror symmetry.
- We'll also deduce parts of closed string mirror symmetry:

$$QH^\bullet(X) \cong H^*(\Lambda^* T\check{\mathcal{X}});$$

counts of rational curves in $X =$ periods of volume form on $\check{\mathcal{X}}$.

A-side: symplectic set-up

- An integral symplectic CY manifold consists of
  - a compact manifold X^{2n};
  - a symplectic form ω on X;
  - a codimension 2 symplectic submanifold $D \subset X$;
  - a 1-form $\theta \in \Omega^1(X \setminus D)$ such that $d\theta = \omega|_{X \setminus D}$; and
  - Nowhere-vanishing complex volume form $\Omega \in C^\infty(X; K_X)$.
- Main class of examples:
  - X complex manifold;
  - ω curvature of a hermitian holomorphic line bundle;
  - $D = s^{-1}(0)$ where s is a holomorphic section;
  - θ the connection in the trivialization s;
  - Ω a holomorphic volume form.
- Warning: we only expect to form a mirror to X when it's complex and on the brink of degenerating completely!

B-side: algebraic set-up

- Laurent series field $\mathbb{K} = \mathbb{C}[q^{-1}][\![q]\!]$.
- Think of $\operatorname{Spec} \mathbb{K}$ as an algebraic 'punctured disc': home for Laurent expansions of meromorphic functions on $\Delta^*(r)$.
- Our B-side CY varieties will be smooth n-dimensional projective varieties $\check{\mathfrak{X}}$ over $\mathbb{K}$, with 'holomorphic volume forms'

$$\check{\Omega} \in H^0(\Lambda^n T^* \check{\mathfrak{X}}).$$

- Were $\check{\mathfrak{X}}$ defined by homogeneous polynomials whose coefficients (in $\mathbb{K}$) had positive radius of convergence, we could turn $\check{\mathfrak{X}}$ into a holomorphic family $\check{\mathbb{X}} \to \Delta^*(r)$ over a punctured disc.

  E.g.

$$\{Y^2 Z = X^3 + a(q) X Z^2 + b(q) Z^3\} \subset \Delta^*(r) \times \mathbb{C}P^2.$$

  We do not want to assume convergence.

B-side: quasi-unipotent monodromy

- Algebraic de Rham cohomology

$$H^\bullet_{DR}(\check{\mathfrak{X}}/\mathbb{K}) = \mathbb{H}^*(\Omega^*_{\check{\mathfrak{X}}/\operatorname{Spec}\mathbb{K}}) \quad \text{(graded } \mathbb{K}\text{-algebra)}$$

  carries an automorphism $T \in \operatorname{Aut} H^\bullet_{DR}(\check{\mathfrak{X}}/\mathbb{K})$: the monodromy of the Gauss–Manin connection $\nabla_{d/dq}$.
- If $\check{\mathfrak{X}}$ is the Laurent expansion of a holomorphic family with fibers $\check{X}_q$, T can be identified with the monodromy around $S^1(r)$ acting in $H^*_{sing}(\check{X}_q; \mathbb{K})$.
- Quasi-unipotency: after a substitution $q \mapsto q^k$, the monodromy is unipotent of exponent $n+1$:

$$(T - I)^{n+1} = 0.$$

  [Griffiths–Landman–Grothendieck; N. Katz]

Maximally unipotent monodromy

- We assume that the monodromy T of $\check{\mathfrak{X}}$ is maximally unipotent:

$$(T - I)^{n+1} = 0, \quad (T - I)^n \neq 0.$$

- Maximal unipotency means that $\check{\mathfrak{X}}$ is a punctured disc around a point in the deepest stratum of CY moduli space.
- *Example:* a CY hypersurface in projective space degenerating to a union of hyperplanes.
- Maximal unipotency is a reasonable assumption: it is needed for $\check{\mathfrak{X}}$ to have a (closed-string) mirror X.

A-side: closed string invariants

- The small quantum cohomology $QH^*(X) = H^*(X;\mathbb{K})$ is a graded $\mathbb{K}$-algebra under the quantum product $\star$ (associative, graded, unital, graded-commutative).
- Structure constants of $\star$ 'count' pseudo-holomorphic spheres $u\colon S^2 \to X$ weighted as $q^{u\cdot D}$.
- Integration

$$\int_X : QH^{2n}(X) \to \mathbb{K}$$

makes $QH^*(X)$ into a Frobenius algebra. That is, $QH^i(X)$ is perfectly paired with $QH^{2n-i}(X)$ via

$$(a,b) \mapsto \int_X a * b.$$

B-side: closed string invariants

- The tangential cohomology

$$HT^*(\check{\mathfrak{X}}) = \bigoplus_{p+q=*} H^p(\Lambda^q T\check{\mathfrak{X}})$$

is also graded $\mathbb{K}$-algebra.

- The volume form $\check{\Omega} \in H^0(\Lambda^n T^*\check{\mathfrak{X}})$ determines a trace map

$$\mathrm{tr}\colon HT^{2n}(\check{\mathfrak{X}}) \to \mathbb{K}$$

making $HT^*(\check{\mathfrak{X}})$ a Frobenius algebra.

- In complex-analytic terms, represent $\eta(q) \in H^n(\Lambda^n T\check{X}_q)$ by $\eta(q) \in C^\infty(\Lambda^{0,n}T^* \otimes \Lambda^n T)$. Contract $\eta(q)$ with $\check{\Omega}_q$ to get a $(0,n)$-form $\iota(\eta)\check{\Omega}$. Put

$$\mathrm{tr}\,\eta(q) = \int_{\check{X}_q} \check{\Omega}_q \wedge \iota(\eta)\check{\Omega}_q.$$

Distinguished degree 2 classes

A-side

- symplectic class $D \in QH^2(X)$.
- $D^{*n} = D^{\smile n} + O(q)$ is non-zero.

B-side

- Kodaira–Spencer class $\theta \in H^1(T\check{\mathfrak{X}}) \subset HT^2(\check{\mathfrak{X}})$ for the vector field $q\frac{d}{dq}$ on the punctured disc.
  If $\check{\mathfrak{X}}$ is the Laurent expansion of a map from a punctured disc to CY moduli space then θ is the derivative of this map.
- Maximal degeneration assumption:

$$\theta^n \neq 0 \in H^n(\Lambda^n T\check{\mathfrak{X}})$$

Compelx analytic case: maximally unipotent monodromy $\Rightarrow$ maximal degeneration. Proof uses mixed Hodge theory [Deligne; Schmid]

A-side: open string invariants

- The Fukaya category for X relative to D is a $\mathbb{K}$-linear A_∞ category. Its objects are closed, exact Lagrangian submanifolds [with gradings and pin structures].

$$L^n \subset X^{2n} \setminus D: \quad \theta|_L = d(\text{some function on } L)$$

- Morphism space $\hom(L_0, L_1)$ is Floer's cochain space

$$CF(L_0, L_1) = \mathbb{K}^{L_0 \cap L_1}$$

  when $L_0 \pitchfork L_1$.
- A_∞ operations μ^d, $d \geq 0$, count pseudo-holomorphic $(d+1)$-gons u in X, bounded by exact Lagrangians, weighted by $q^{u \cdot D}$.
- Working only with exact Lagrangians in $X \setminus D$ results in a large saving in foundational complexity [Sheridan].

B-side: open string invariants

- Work with a DG model Perf $\check{\mathcal{X}}$ for the derived category.
- A DG category is the same thing as an A_∞-category with vanishing higher compositions μ^d $(d \geq 3)$.
- Objects: finite complexes of algebraic vector bundles.
- Morphism spaces: Čech cochain complexes

$$\check{C}^\bullet(\mathcal{U}; \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F}))$$

  with respect to a fixed open affine cover $\mathcal{U}$.
- The cohomology of the hom-space is $\mathrm{RHom}^\bullet(\mathcal{E}, \mathcal{F})$, the derived sheaf homomorphisms from $\mathcal{E}$ to $\mathcal{F}$.

Weak CY structures

- A weak CYn structure on an A_∞-category $\mathcal{C}$ is a quasi-isomorphism of $(\mathcal{C}, \mathcal{C})$-bimodules

$$\beta\colon \mathcal{C} \to \mathcal{C}^\vee[n] \quad \text{such that} \quad \beta^\vee \simeq \beta.$$

  Non-degenerate symmetric bilinear form on a category.
- $\mathcal{F}(X, D)$ has an intrinsic weak CYn structure:

$$HF^*(L_0, L_1) \cong HF^{n-*}(L_1, L_0)^\vee.$$

- Serre duality and the volume form $\check{\Omega}$ determine a CYn structure on Perf $\check{\mathfrak{X}}$:

$$\mathrm{RHom}^*(\mathcal{E}, \mathcal{F}) \cong \mathrm{RHom}^{n-*}(\mathcal{F}, \mathcal{E})^\vee.$$

Homological mirror symmetry (HMS)

- Mirror symmetry identifies mirror pairs of CY varieties. Some are relatively simple (e.g. the mirror to the quintic 3-fold). Some are very sophisticated (e.g. Gross–Siebert program).
- A version of Kontsevich's HMS conjecture predicts a quasi-equivalence

$$\psi\colon \underline{\mathcal{F}(X, D)} \to \mathrm{Perf}\, \check{\mathfrak{X}}$$

of $\mathbb{K}$-linear weak CYn A_∞-categories.
- The underline denotes a certain algebraic enlargement of $\mathcal{F}(X, D)$ that I'm not going to explain.

Weak HMS

- We shall assume *weak homological mirror symmetry*. That is, we suppose given
  (i) Some Lagrangians forming a full subcategory $\mathcal{A} \subset \mathcal{F}(X, D)$;
  (ii) Some perfect complexes a full subcategory $\mathcal{B} \subset \operatorname{Perf} \check{\mathcal{X}}$ which split-generates $\operatorname{Perf} \check{\mathcal{X}}$;
  (iii) an A_∞-functor
  $$\psi \colon \mathcal{A} \to \mathcal{B},$$
  respecting weak CYn structures, such that
  $$H^*\psi \colon H^*\mathcal{A} \to H^*\mathcal{B}$$
  is a categorical isomorphism.
- Split generation means that the closure of $\mathcal{B}$ is $\operatorname{Perf} \check{\mathcal{X}}$ under the following operations: shifts, mapping cones, isomorphisms, passing to direct summands.
- One could take $\mathcal{B} = \{\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots\}$.

Statement of results

(X^{2n}, D): integral symplectic CY $2n$-manifold.
$(\check{\mathfrak{X}}, \check{\Omega})$: smooth, projective $\mathbb{K}$-variety, max. degenerate $(\theta^n \neq 0)$
e.g. analytic family with maximally unipotent monodromy.

Theorem (Sheridan–P.)

Assume weak HMS, i.e., $\psi \colon \mathcal{A} \xrightarrow{\simeq} \mathcal{B}$, *where* $\mathcal{B}$ *split-generates* $\mathrm{Perf}\, \check{\mathfrak{X}}$. *Then*

1. $\mathcal{A}$ *split-generates* $\underline{\mathcal{F}(X, D)}$, *and so full HMS holds.*
2. ψ *determines an isomorphism of graded* $\mathbb{K}$*-algebras*

$$\kappa \colon QH^*(X) \to HT^*(\check{\mathfrak{X}})$$

   preserving the distinguished degree 2 elements: $\kappa(D) = \theta$.
3. ψ *also preserves Frobenius traces, and consequently*

$$\int_X D^{*n} = \operatorname{tr} \theta^n.$$

Caveat

We have not (yet) proved that the volume form $\check{\Omega}$ has the standard form demanded by Hodge-theoretic mirror symmetry. Thus the enumerative formula

$$\int_X D^{*n} = \operatorname{tr} \theta^n = \int_{\check{X}_q} \check{\Omega}_q \wedge \iota(\theta^n) \check{\Omega}_q$$

is not yet in practical form for counting curves.

Proof that $\mathcal{A}$ split-generates $\mathcal{F}(X, D)$: A-side

- The closed-open string map is a map of rings

$$CO|_{\mathcal{A}} \colon QH^*(X) \to HH^*(\mathcal{A})$$

  whose target is Hochschild cohomology
  (= natural transformations $\mathrm{id}_{\mathcal{A}} \Rightarrow \mathrm{id}_{\mathcal{A}}$).
- Abouzaid's generation criterion for Fukaya categories
  dual version [Abouzaid–Fukaya–Oh–Ohta–Ono]:
  $CO|_{\mathcal{A}}$ injective in top degree $2n \Rightarrow \mathcal{A}$ split-generates $\mathcal{F}(X, D)$
- $QH^{2n}(X)$ is generated by $D^{\star n}$, so we want $CO|_{\mathcal{A}}(D^{\star n}) \neq 0$.
- Sheridan earlier observed that

$$CO(D) = \left[q \frac{d\mu^{\bullet}_{\mathcal{A}}}{dq} \right] \in HH^2(\mathcal{A}).$$

Proof that $\mathcal{A}$ split-generates $\mathcal{F}(X, D)$: B-side

- The *twisted Hochschild–Kostant–Rosenberg map* is a ring isomorphism

$$\mathrm{HKR}\colon HT^*(X) \to HH^*(\mathrm{Perf}\,\check{\mathcal{X}}).$$

[..., Calaque–van den Bergh–Rossi]

- $\mathrm{HKR}(\theta)$ is the categorical analog of the Kodaira–Spencer class, describing how $\mathrm{Perf}\,\check{\mathcal{X}}$ varies in q.
- Since $\mathcal{B}$ split-generates $\mathrm{Perf}\,\check{\mathcal{X}}$ by assumption,

$$HH^*(\mathrm{Perf}\,\check{\mathcal{X}}) = HH^*(\mathcal{B}).$$

Putting together A and B

- Given weak HMS, we have a composite ring map κ:

$$\begin{array}{ccccc} QH^*(X) & \xrightarrow{CO|_{\mathcal{A}}} & HH^*(\mathcal{A}) & & \\ \downarrow \kappa & & \downarrow HH^*(\psi) & & \\ HT^*(\check{\mathfrak{X}}) & \xleftarrow{\mathrm{HKR}^{-1}} & HH^*(\mathrm{Perf}\,\check{\mathfrak{X}}) & = & HH^*(\mathcal{B}) \end{array}$$

- Under these maps:
  $D \mapsto [q(d\mu/dq)] \mapsto$ categorical KS $\mapsto$ geometric KS.
- That is, $\kappa(D) = \theta$. So $\kappa(D^{\star n}) = \theta^n \neq 0$. Abouzaid's criterion tells us that $\mathcal{A}$ split-generates.

Isomorphism of quantum and tangential cohomology

- We now have
$$\kappa\colon QH^*(X) \to HT^*(\check{\mathfrak{X}})$$
ring map, $\kappa(D^{\star n}) \neq 0$.
- By Poincaré duality for $QH^*(X)$, $CO|_{\mathcal{A}}$ is injective.
- From this it follows that $CO|_{\mathcal{A}}$ is surjective [Ganatra]. So $CO|_{\mathcal{A}}$ is an isomorphism.
- Hence κ is a ring isomorphism.
- Key new ingredients: use Abouzaid's criterion; role of maximal degenerations; recent advances on HKR. Couple these with homological algebra.