Quantum footprints of symplectic rigidity

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- quantum speed limit (with Laurent Charles)
- noise-localization uncertainty (recent developments)

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Example: For pure states $\xi, \eta \in H$, $|\xi| = |\eta| = 1$,
$\Phi(\xi, \eta) = |\langle \xi, \eta \rangle|$.

Dislocation

$F_t \in \mathcal{L}(H)$ - quantum Hamiltonian.
Schrödinger equation
$\dot{U}_t = -\frac{i}{\hbar} F_t U_t$,
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Appears e.g. in quantum computation. Margolus-Levitin (1998) address the question about *"the maximum number of distinct [i.e., non-overlapping] states that the system can pass through, per unit of time. For a classical computer, this would correspond to the maximum number of operations per second."*

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Quantum speed limit: universal bound on the energy required to a-dislocate a quantum state:

$$\Phi(\theta, U\theta U^{-1}) \leq a \;\Rightarrow\; \ell_q(F) \geq \arccos(a)\hbar$$

Mandelstam-Tamm, 1945 "time-energy uncertainty", Uhlmann 1992, Margolus-Levitin, 1998

Quantum speed limit

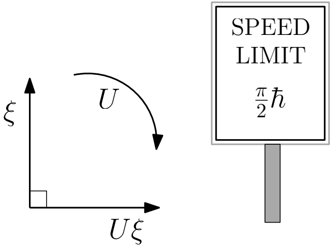

Figure: "Displacing" a pure quantum state

We explore semiclassical dislocation of semiclassical states.

Displacement energy

(M, ω) - closed symplectic manifold.
Let f_t, $t \in [0, 1]$ be classical Hamiltonian generating Hamiltonian diffeomorphism $\varphi \in Ham(M, \omega)$. Total energy

$\ell_c(f) = \int_0^1 ||f_t|| dt$, where $||g|| := \max |g|$-uniform norm.

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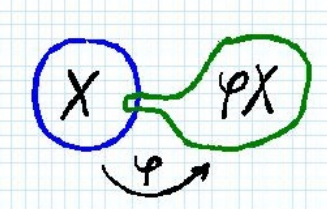
Rigidity: $e(X) > 0$ for all open X

Counterpoint: If $\text{Vol}(X) < \frac{1}{2} \cdot \text{Vol}(M)$, for all $\epsilon > 0, \delta > 0$ there exists f_t such that

$\text{Vol}(\varphi X \cap X) < \epsilon$, $\ell_c(f) < \delta$.



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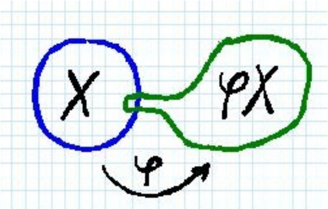

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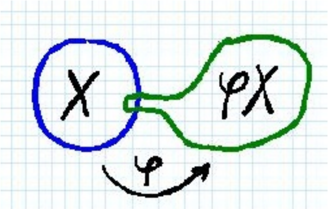

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Based on Katok's lemma, 1970, Ostrover-Wagner, 2005.

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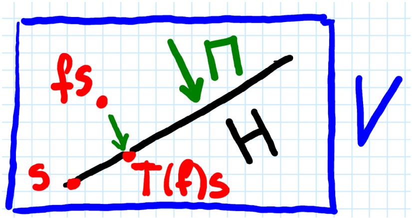
The Toeplitz operator: $T_\hbar(f)(s) := \Pi_\hbar(fs)$, $f \in C^\infty(M)$, $s \in H_\hbar$.



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Definition: For classical state τ (probability measure on M)

$$Q_\hbar(\tau) = \int_M P_{x,\hbar} d\tau(x) \in \mathcal{S}(H_\hbar)$$

"classical" quantum state, Giraud-Braun-Braun 2008

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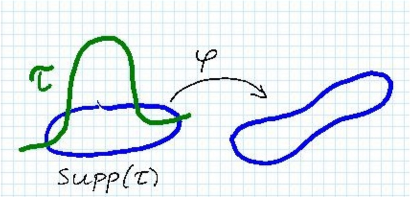
Theorem (Charles-P., 2016)

If f_t displaces supp(τ) $\Rightarrow$ F_t $O(\hbar^\infty)$-dislocates θ.

Figure: φ-time-one-map of the flow of f_t



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Conclusion: Speed limit becomes more restrictive ~ 1 than the universal bound $\sim \hbar$.

Rigidity vs. flexibility on quantum side

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Assume $Vol(supp(\tau)) < \frac{1}{2} \cdot Vol(M)$. *Then* $\forall \epsilon, \delta > 0$ *there exists* f_t *such that* F_t ϵ*-dislocates* θ *and* $\ell_q(F_t) < \delta$.

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	RIGIDITY	FLEXIBILITY
RATE OF DISLOCATION	$o(\hbar^n)$	ϵ

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(P4) **(trace correspondence)**
$\left|\text{trace}(T_\hbar(f)) - (2\pi\hbar)^{-n}\int_M f\frac{\omega^n}{n!}\right| = O(\hbar^{-(n-1)})$,

for all $f, g \in C^\infty(M)$.

Zooming into small scales

Theorems 1,2 extend to dislocation of semiclassical states which "occupy" a ball of radius $\hbar^{\varepsilon}$, $\varepsilon \in [0, 1/2)$ in the phase space. The speed limit on such a scale is $\sim \hbar^{2\varepsilon}$ which, again, is more restrictive than the universal quantum speed limit $\sim \hbar$.

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Rigidity of remainders: (Charles-P., 2016) α, β, γ cannot be small simultaneously

Quantum "proof" of $e(B) > 0$

$B(r) \subset \mathbb{R}^{2n}$ – ball of radius r. To prove: $e(B(r)) \gtrsim r^2$.
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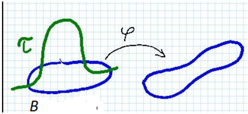

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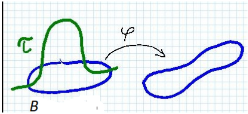

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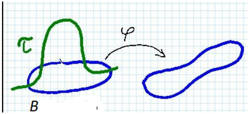

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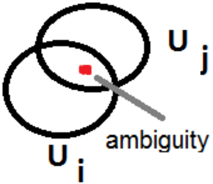
Resolution: Remainders of quantization are large on scale $\sim \sqrt{\hbar}$

Phase space localization

$\mathcal{U} = \{U_j\}$- open cover of (M, ω)

Classical: Register $z \in M$ in exactly one $U_j \ni z$. Ambiguity because of overlaps.



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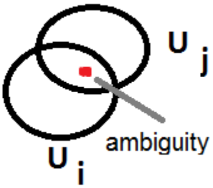

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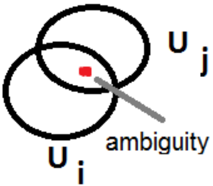

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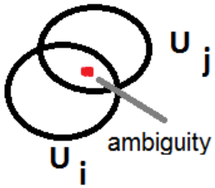

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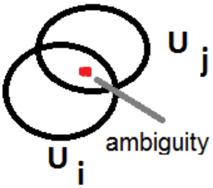

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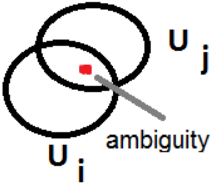

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Fine localization $\Rightarrow$ large noise.

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$\mathcal{U} = \{U_j\}_{j=1}^N$- open cover by displaceable sets
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Conjecture (P., counterpart of noise-localization)

$pb(\mathcal{U}) \cdot e(\mathcal{U}) \geq C(M, \omega)\ \forall \mathcal{U}$.

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Entov-P, 2006 Open cover $\mathcal{U}$ by displaceable open sets does not admit a Poisson commuting partition of unity.

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Buhovsky-Shira Tanny-Logunov (2018): Proof for closed surfaces with universal C . (cf. Jordan Payette, 2018).

The End

THANK YOU!