Joint equidistribution of adelic torus orbits and families of twisted L-functions

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First Linnik problem

For $D \in \mathbb{N}$ put $\mathscr{R}_D = \{(x,y,z) \in \mathbb{Z}^3_{\mathrm{prim}} : x^2+y^2+z^2 = D\}$

Legendre: $\mathscr{R}_D \neq \emptyset$ iff $D \in \mathbb{D}$, where $\mathbb{D} = \{D \not\equiv 0,4,7 \mod 8\}$.

Gauss, Siegel, Dirichlet: for $D \in \mathbb{D}$: $|\mathscr{R}_D| = D^{1/2+o(1)}$.

Write

$$\mathscr{S}_D = \left\{ \frac{v}{\|v\|} : v \in \mathscr{R}_D \right\} \subset S^2 = \{x^2+y^2+z^2 = 1\}.$$

Let μ_{S^2} be the normalized Lebesgue measure on the sphere S^2.

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Conjecture A: *Equidistribution of integer points on the sphere*

For $D \in \mathbb{D}$ let

$$\mu_{\mathscr{S}_D} = \frac{1}{|\mathscr{S}_D|} \sum_{u \in \mathscr{S}_D} \delta_u.$$

Then $\mu_{\mathscr{S}_D}$ weak-* converges to μ_{S^2} as $D \to \infty$ in $\mathbb{D}$.

The coronavirus

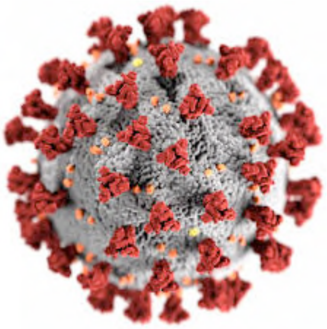

Figure: Covid-19

Numerical example

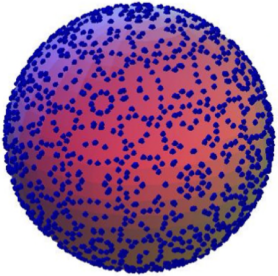

Figure: Integer points of norm 104851 projected onto S^2

Ellenberg, Michel, Venkatesh, *Linnik's ergodic method and the distribution of integer points on spheres*

Linnik (1950-60's)

Let $p > 2$ be prime and write

$$\mathbb{D}(p) = \{D \in \mathbb{D} : -D \in (\mathbb{F}_p^\times)^2\}.$$

Then $\mu_{\mathscr{S}_D} \xrightarrow{w^*} \mu_{S^2}$ as $D \to \infty$ in $\mathbb{D}(p)$.

Linnik's condition $D \in \mathbb{D}(p)$ is equivalent to

the stabilizer of a point in $\mathscr{R}_D$ is a split torus over $\mathbb{Q}_p$.

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Basic idea of ergodic method: let ν be a weak-* limit.

1. Show that ν has maximal entropy, by bootstrapping an upper bound on the spacing of nearby points (Linnik's Basic Lemma)
2. Apply uniqueness result of Einsiedler–Lindenstrauss.

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2. Apply uniqueness result of Einsiedler–Lindenstrauss.

Quantitative version: $\mu_{\mathscr{S}_D} \xrightarrow{w^*} \mu_{S^2}$ as $D \to \infty$ in the set

$$\left\{D \in \mathbb{D} : \exists\ p \ll D^{\frac{1}{o(\log\log D)}} \text{ with } p \text{ split in } \mathbb{Q}(\sqrt{-D})\right\}.$$

This set is all of $\mathbb{D}$ under GRH!

Golubeva–Fomenko (1987), following Iwaniec (1987)

Conjecture A is true with a power savings rate: there is $\delta > 0$ such that for every "nice" $\Omega \subset S^2$ we have

$$\mu_{\mathscr{S}_D}(\Omega) = \mu_{S^2}(\Omega) + O(D^{-\delta}).$$

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Spectral (automorphic) method: bound Weyl sums

$$W(f, D) = \frac{1}{|\mathscr{S}_D|} \sum_{u \in \mathscr{S}_D} f(u),$$

where $f \in C(S^2)$ and $\langle f, 1 \rangle = 0$.

Enough to test on an orthonormal basis of $L^2_0(S^2)$.

We take an orthonormal basis of arithmetic eigenfunctions. Recall

$$S^2 = \mathrm{SO}(3)/\mathrm{SO}(2), \quad \mathrm{SO}(3) = \boldsymbol{H}^\times/\mathbb{R}^\times = \boldsymbol{G}(\mathbb{R}),$$

where $\boldsymbol{G} = \boldsymbol{P}\boldsymbol{B}^\times$ and $\boldsymbol{B} = \boldsymbol{B}^{(2,\infty)}$.

Let $\Gamma = \boldsymbol{G}(\mathbb{Z})$. Then Γ acts on $\mathscr{R}_D$ by conjugation. Thus

$$W(f, D) = \frac{1}{|\Gamma \backslash \mathscr{S}_D|} \sum_{u \in \Gamma \backslash \mathscr{S}_D} F(u),$$

where

$$F(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} F(\gamma x) \quad \text{on} \quad \boldsymbol{S}^2 = \Gamma \backslash S^2.$$

We have

$$\boldsymbol{S}^2 = \boldsymbol{G}(\mathbb{Q}) \backslash \boldsymbol{G}(\mathbb{A}) / \boldsymbol{G}(\hat{\mathbb{Z}}) \mathrm{SO}(2).$$

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We have

$$\boldsymbol{S}^2 = \boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})/\boldsymbol{G}(\hat{\mathbb{Z}})\mathrm{SO}(2).$$

Take o.n.b. $\{f_i\}$ of $L_0^2(S^2)$ consisting of spherical harmonics

$$\Delta_{S^2} f_i = k(k+1) f_i, \qquad k \geq 1,$$

such that, upon adelization, the φ_i on $\boldsymbol{S}^2$ are joint eigenfunctions of the Hecke algebra $\mathcal{H}(\boldsymbol{G}(\hat{\mathbb{Z}})\backslash\boldsymbol{G}(\mathbb{A}_f)/\boldsymbol{G}(\hat{\mathbb{Z}}))$.

Moreover, $\Gamma\backslash\mathscr{R}_D$ is a torsor for the class group C_D of (an order in) $\mathbb{Q}(\sqrt{-D})$. Fixing a base point $u \in \mathscr{S}_D$ we have

$$W(f,D) = \frac{1}{h(-D)} \sum_{t \in C_D} F(t.u),$$

where $h(-D) = |C_D|$. This is an adelic toric integral: let

$$\boldsymbol{T}_D = (\mathrm{Res}_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}} \mathbb{G}_m)/\mathbb{G}_m.$$

Choosing $u \in \mathscr{S}_D$ yields an embedding $\boldsymbol{T}_D \hookrightarrow \boldsymbol{G}$. Let

$$\boldsymbol{T}_D(\hat{\mathbb{Z}}) = \boldsymbol{T}_D(\mathbb{A}_f) \cap \boldsymbol{G}(\hat{\mathbb{Z}}) \quad \text{and} \quad T(\mathbb{R}) = g_\infty^{-1}\mathrm{SO}(2) g_\infty.$$

Get an adelic toric orbit (finite collection of points)

$$Z_D = \boldsymbol{T}_D(\mathbb{Q})\backslash \boldsymbol{T}_D(\mathbb{A}) g_\infty / \boldsymbol{T}_D(\hat{\mathbb{Z}}) \boldsymbol{T}(\mathbb{R}) \hookrightarrow \boldsymbol{S}^2.$$

Then $W(f;D) = \frac{1}{h(-D)} \int_{Z_D} \varphi$, where φ is the adelization of F.

Waldspurger (1985) *et al.*

Let $\sigma = \langle \varphi \rangle$ on $\boldsymbol{G} = \boldsymbol{PB}^{\times}$. Let $\pi = \mathrm{JL}(\sigma)$ on PGL_2. Then

$$|W(f;D)|^2 \doteq D^{-1/2} \frac{L(1/2,\pi)L(1/2,\pi\times\eta_D)}{L(1,\eta_D)L(1,\mathrm{Ad}\,\pi)}.$$

Remark: If f is of degree $k \geq 1$ then $\sigma_\infty = \mathrm{sym}^{2k}$ on $\boldsymbol{G}(\mathbb{R}) = \mathrm{SO}(3)$ and $\pi_\infty = \mathrm{JL}(\sigma_\infty) = D_{2k+2}$ on $\mathrm{PGL}_2(\mathbb{R})$.

Siegel bound: We have $L(1,\eta_D) \gg_\epsilon D^{-\epsilon}$.

The problem is reduced to subconvex bounds on twists of L-functions by (quadratic) Dirichlet character twists.

Duke–Friedlander–Iwaniec (1993)

There is $\delta > 0$ such that $L(1/2,\pi\times\eta_D) \ll D^{1/2-\delta}$.

Second Linnik problem

For $D \in \mathbb{N}$ with $D \equiv 0, 3 \pmod 4$ let

$\mathcal{Q}_D = \{AX^2 + BXY + CY^2 : \text{primitive}, \ B^2 - 4AC = -D\}/\mathrm{SL}_2(\mathbb{Z})$.

Let $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ be the modular surface.

Definition

Put $\mathscr{H}_D = \{\text{unique root of } Q(X,1) \text{ in } \mathbb{H} : Q \in \mathcal{Q}_D\} \subset Y(1)$.

Let $\mu_{Y(1)}$ be the normalized hyperbolic measure on $Y(1)$.

Conjecture B: *Equidistribution of Heegner points on $Y(1)$*

For $D \in \mathbb{N}$ with $D \equiv 0, 3 \pmod 4$ let

$$\mu_{\mathscr{H}_D} = \frac{1}{|\mathscr{H}_D|} \sum_{z \in \mathscr{H}_D} \delta_z.$$

Then $\mu_{\mathscr{H}_D}$ weak-* converges to $\mu_{Y(1)}$ along $D \equiv 0, 3 \pmod 4$.

Linnik (1950's-60's)

Fix $p > 2$ a prime. Then $\mu_D \xrightarrow{w^*} \mu_{Y(1)}$ along $D \equiv 0, 3 \pmod 4$ such that $-D \in (\mathbb{F}_p^\times)^2$.

Again, a quantitative version leads to Conjecture B under GRH.

Duke (1988)

Conjecture B holds unconditionally, with a power savings rate.

Same ideas: Weyl sums $\to$ Waldspurger $\to$ Subconvexity

For $f \in C_0^\infty(Y(1))$ we wish to write the normalized Weyl sum

$$W(f; D) = \frac{1}{|\mathscr{H}_D|} \sum_{z \in \mathscr{H}_D} f(z) = \frac{1}{|h(-D)|} \sum_{t \in C_D} f(t.z_0)$$

where $z_0 \in \mathscr{H}_D$, as an adelic torus integral. We have

$$Y(1) = \mathrm{PGL}_2(\mathbb{Q}) \backslash \mathrm{PGL}_2(\mathbb{A}) / \mathrm{PGL}_2(\hat{\mathbb{Z}})\mathrm{SO}(2).$$

From $z_0 \in \mathscr{H}_D$ get embedding

$$\boldsymbol{T}_D = (\mathrm{Res}_{\mathbb{Q}(\sqrt{-D})/\mathbb{Q}} \mathbb{G}_m)/\mathbb{G}_m \hookrightarrow \mathrm{PGL}_2.$$

Let $\boldsymbol{T}_D(\hat{\mathbb{Z}}) = \boldsymbol{T}_D(\mathbb{A}_f) \cap \mathrm{PGL}_2(\hat{\mathbb{Z}})$ and $T(\mathbb{R}) = g_\infty^{-1} \mathrm{SO}(2) g_\infty$. Get

$$Z_D = \boldsymbol{T}_D(\mathbb{Q}) \backslash \boldsymbol{T}_D(\mathbb{A}) g_\infty / \boldsymbol{T}_D(\hat{\mathbb{Z}}) \boldsymbol{T}(\mathbb{R}) \hookrightarrow Y(1).$$

Then $W(f; D) = \frac{1}{h(-D)} \int_{Z_D} \varphi$, where φ is the adelization of f.

Waldspurger (1985) *et al.*

Let $\pi = \langle \varphi \rangle$ be cuspidal Maass on PGL_2. Then

$$|W(f; D)|^2 \doteq D^{-1/2} \frac{L(1/2, \pi) L(1/2, \pi \times \eta_D)}{L(1, \eta_D)^2 L(1, \mathrm{Ad}\,\pi)}.$$

Remark: Here π_∞ is a principal series representation on $\mathrm{PGL}_2(\mathbb{R})$.

The same subconvexity bound of DFI (1993) solves the problem.

Other variants

1) *Sparse equidistribution*: twisted Weyl sums, the numerator becomes $L(1/2, \pi \times \pi_\chi)$, subconvex bounds by Michel (2004)

2) Let $\mathbb{Q}(\sqrt{D})$ be real quadratic. Then $\boldsymbol{T}_D \hookrightarrow \boldsymbol{PB}^\times$ for any *indefinite* $\boldsymbol{B}$ such that p split in $\mathbb{Q}(\sqrt{D})$ implies $\boldsymbol{B}(\mathbb{Q}_p)$ split.

Obtain packets of closed geodesics on the unit tangent bundle of Shimura or modular curves.

Skubenko (1950-60's)

Equidistribution under Linnik's condition.

Duke (1987)

Equidistribution for all positive fundamental discriminants.

Both proofs follow the same pattern.

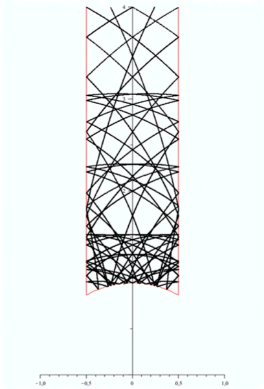

Figure: $h(\mathbb{Q}(\sqrt{377})) = 1$

Einsiedler–Lindenstrauss–Michel–Venkatesh, *The distribution of closed geodesics on the modular surface, and Duke's theorem*. Ergodic proof without congruence conditions! (torus split at ∞)

2) Let $\boldsymbol{G} = \boldsymbol{PB}^{\times}$, where $B = B^{(p,\infty)}$, where $p > 2$. Then

$$\boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})/\boldsymbol{G}(\hat{\mathbb{Z}})\boldsymbol{G}(\mathbb{R}) \simeq \mathrm{Ell}_p^{\mathrm{ss}},$$

which has size $\frac{p-1}{12} + O(1)$ and has a natural probability measure

$$\mu_{\mathrm{Ell}_p^{\mathrm{ss}}}(e) = \frac{|\mathrm{Aut}(e)|^{-1}}{\sum_{e' \in \mathrm{Ell}_p^{\mathrm{ss}}} |\mathrm{Aut}(e')|^{-1}} = \frac{12}{p-1}|\mathrm{Aut}(e)|^{-1}.$$

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Let F be an imaginary quad field for which p is inert. Let H_F be the Hilbert class field of F. For $\mathfrak{p} \mid p$ get reduction map

$$\mathrm{Ell}_{\mathcal{O}_F}^{\mathrm{cm}} \to \mathrm{Ell}_p^{\mathrm{ss}}, \qquad E \mapsto E \bmod \mathfrak{p}.$$

See Aka–Luethi–Michel–Wieser (2020).

Michel (2004)

Then the fibers of the reduction map are distributed according to $\mu_{\mathrm{Ell}_p^{\mathrm{ss}}}$ as F varies over IQF for which p is inert, with power savings in the disciminant.

Simultaneous equidistribution

We return to the Linnik problems A and B.

Let $\boldsymbol{G}_1 = \boldsymbol{PB}^\times$, where $\boldsymbol{B} = \boldsymbol{B}^{(2,\infty)}$, and $\boldsymbol{G}_2 = \mathrm{PGL}_2$. For $D \in \mathbb{D}$:

$$\boldsymbol{G}_1 \hookleftarrow \boldsymbol{T}_D \hookrightarrow \boldsymbol{G}_2,$$

simultaneous embeddings. We can then construct

$$\Delta : \boldsymbol{T}_D \hookrightarrow \boldsymbol{G}_1 \times \boldsymbol{G}_2, \qquad \Delta : Z_D \hookrightarrow \boldsymbol{S}^2 \times Y(1).$$

Understand the distribution of ΔZ_D inside $\boldsymbol{S}^2 \times Y(1)$ as $D \to \infty$.

Expectation

ΔZ_D should equidistribute to $\boldsymbol{S}^2 \times Y(1)$ since the spaces $\boldsymbol{S}^2$ and $Y(1)$ come from non-isomorphic quaternion algebras.

Classical description of ΔZ_D

We have

$$Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) = \mathcal{L}_2,$$

where $\mathcal{L}_2$ is the space of unimodular lattices in $\mathbb{R}^2$ up to rotation.

Let $D \in \mathbb{D}$. For $v \in \mathscr{R}_D$ consider $\Lambda_v = \mathbb{Z}^3 \cap v^\perp$. Then

- rotate to a reference plane in $\mathbb{R}^3$,
- normalize to have covolume 1.

We obtain $[\Lambda_v] \in \mathcal{L}_2$.

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- rotate to a reference plane in $\mathbb{R}^3$,
- normalize to have covolume 1.

We obtain $[\Lambda_v] \in \mathcal{L}_2$. Then

$$Z_D = \left\{ \left(\frac{v}{\|v\|}, [\Lambda_v] \right) : v \in \mathscr{R}(D) \right\} \subset S^2 \times Y(1).$$

So the question becomes:

Does a primite integral point on the sphere and the shape of its orthogonal lattice equidistribute in $S^2 \times Y(1)$?

Conjecture: Michel–Venkatesh (2006), Aka–Einsiedler–Shapira

ΔZ_D equidistributes to $\mu_{\mathbf{S}^2} \times \mu_{Y(1)}$ as $D \to \infty$ in $\mathbb{D}$.

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Aka–Einsiedler–Shapira (2016)

Let $p, q > 2$ be distinct. Then ΔZ_D equidistributes to $\mu_{\mathbf{S}^2} \times \mu_{Y(1)}$ as $D \to \infty$ in $\mathbb{D}(p, q) \cap \mathbb{F}$, where

$$\mathbb{D}(p, q) = \left\{ D \in \mathbb{D} : -D \in (\mathbb{F}_p^\times)^2, (\mathbb{F}_q^\times)^2 \right\}$$

and $\mathbb{F}$ is the set of square-free integers.

No quantification is available:

- no rate of equidistribution;
- their proof *does not presently allow one* to replace the congruence conditions by GRH.

Idea of proof of AES

Let ν be a weak-* limit.

1. Show that the push forward along both projections equidistributes in its copy.
2. Show, under the Linnik condition $\mathbb{D}(p, q)$, that ν is invariant under $\mathrm{Stab}_{\mathrm{SO}_3(\mathbb{Q}_S)}(v_S)$, where $v_S \in \mathbb{Z}_S^3$ and $S = \{p, q\}$.

   From (1) and (2) it follows that ν is a "joining".
3. Apply Einsiedler–Lindenstrauss (2015):

   a joining of higher rank torus actions is algebraic.

   Since $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are distinct, there is no non-trivial algebraic subgroup containing both $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$.

Comments

- The proof is general and applies to all "hybrid situations":

Aka–Luethi–Michel–Wieser (2020)

Let p_1, p_2, q_1, q_2 be distinct odd primes. The fibers of

$$\mathrm{Ell}^{\mathrm{cm}}_{\mathcal{O}_F} \to \mathrm{Ell}^{\mathrm{ss}}_{p_1} \times \mathrm{Ell}^{\mathrm{ss}}_{p_2}, \quad E \mapsto (E \bmod \mathfrak{p}_1, E \bmod \mathfrak{p}_2)$$

distribute according to $\mu_{\mathrm{Ell}^{\mathrm{ss}}_{p_1}} \times \mu_{\mathrm{Ell}^{\mathrm{ss}}_{p_2}}$ as $D \to +\infty$ in $\mathbb{D}(q_1, q_1) \cap \mathbb{F}$ such that p_1, p_2 are inert in $\mathbb{Q}(\sqrt{-D})$.

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- One can replace 2 copies by n (pairwise non-isomorphic) copies, with a congruence condition for each copy
- For $Y(1) \times Y(1)$ there is also a mixing conjecture of Michel–Venkatesh, solved by Khayutin (2019) for $D \in \mathbb{D}(p, q) \cap \mathbb{F}$ and a Landau–Siegel zero assumption.

Main result: abstract set-up

Let $\boldsymbol{B}_1, \boldsymbol{B}_2/\mathbb{Q}$ be non-isomorphic, non-split, quaternion algebras.

Let $\boldsymbol{G}_i = \boldsymbol{PB}_i^\times$ and $\boldsymbol{G} = \boldsymbol{G}_1 \times \boldsymbol{G}_2$.

Let $\mathcal{O}_i$ be an Eichler order in $\boldsymbol{B}_i(\mathbb{Q})$.

Let $K_f = K_1 \times K_2 \subset \boldsymbol{G}(\mathbb{A}_f)$, where $K_i = \boldsymbol{P}\widehat{\mathcal{O}_i^\times}$.

Write $K = K_f K_\infty$ where $K_\infty = \mathrm{SO}(2) \times \mathrm{SO}(2) \subset \boldsymbol{G}(\mathbb{R})$.

Put $X = \boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})/K$.

Let F_d be a quadratic field extension of $\mathbb{Q}$ of discriminant d, optimally embedded in $\mathcal{O}_i$.

Let $\Delta : \boldsymbol{T}_d = (\mathrm{Res}_{F_d/\mathbb{Q}}\mathbb{G}_m)/\mathbb{G}_m \hookrightarrow \boldsymbol{G}$, the diagonal inclusion.

Let $g_\infty \in \boldsymbol{G}(\mathbb{R})$ satisfy $g_\infty K_\infty g_\infty^{-1} = \Delta\boldsymbol{T}_d(\mathbb{R})$.

Put $\Delta Z_D = \boldsymbol{G}(\mathbb{Q})\Delta\boldsymbol{T}_d(\mathbb{A})gK$, where $g = (1, g_\infty)$.

Main result: statement

Blomer – B. (in preparation)

Assume GRH. Then ΔZ_d equidistributes in X with a logarithmic rate as $|d| \to \infty$: for every "nice" $\Omega \in X$ we have

$$\mu_{\Delta Z_d}(\Omega) = \mu_X(\Omega) + O_\epsilon((\log |d|)^{-1/4+\epsilon}).$$

Our proof goes through the theory of automorphic forms and Waldspurger's theorem.

Plan for the remaining time:

1. describe a previous approach to this problem by R. Zhang;
2. motivate our different approach;
3. sketch our proof.

In the AES variant, the (unnormalized) Weyl sum is

$$S(\omega, \phi; D) = \sum_{\boldsymbol{v} \in \mathbb{Z}^3_{\mathrm{prim}}, \|\boldsymbol{v}\| = D} \omega\left(\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right) \phi(z_{\boldsymbol{v}}),$$

where ω is a spherical harmonic of degree k on S^2 and ϕ is a Maass cusp form or unitary Eisenstein series on $Y(1)$.

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R. Zhang (2015)

Let

$$E(s, g, \omega, \phi) = \sum_{[\gamma] \in \Gamma_\infty \backslash \mathrm{SL}_3(\mathbb{Z})} \omega(k(\gamma g)) \phi(m(\gamma g)) a(\gamma g)^{-s}$$

be the maximal Eisenstein series for $\mathrm{SL}_3(\mathbb{Z})$ induced from ϕ and transforming under $K = \mathrm{SO}(3)$ by ω. Then

$$E(s, e, \omega, \phi) = \sum_{n \geq 1} S(\omega, \phi; n) n^{-s}.$$

Remarks

1. It is not clear from this description how GRH would imply any non-trivial bound on $S(\omega, \phi; D)$.
2. structurally similar to Petridis–Risager–Raulf (2014): QUE for half-integral weight Eisenstein series follows from bounds on coefficients of a double Dirichlet series.

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R. Zhang (2015)

We have
$$\sum_{n \leq X} S(\omega, \phi; n) \ll_\epsilon X^{\frac{15}{14}+\epsilon}.$$

Want to prove $S(\omega, \phi; D) = o(h(-D))$. This does not imply any bound on $S(\omega, \phi; D)$: they could exhibit cancellation on average.

Note that $\boldsymbol{T}_d \subset \boldsymbol{G}_1$ and $\boldsymbol{T}_d \subset \boldsymbol{G}_2$ are *Strong Gelfand pairs*

$$\forall\ \chi_p \in \widehat{\boldsymbol{T}_d(\mathbb{Q}_p)} : \dim \mathrm{Hom}_{\boldsymbol{T}_d(\mathbb{Q}_p)}(\sigma_p, \chi_p) \leq 1.$$

This *multiplicity one* result lies at the heart of Waldspurger's formula, in which the toric period squared is a *single L-function*.

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But we have the following *Gelfand formation*:

$$\begin{array}{c} \boldsymbol{G}_1 \times \boldsymbol{G}_2 \\ | \\ \boldsymbol{T}_d \times \boldsymbol{T}_d \\ | \\ \Delta \boldsymbol{T}_d \end{array}$$

From which we expect to find a *family of L-functions*.

Take F_d IQF, C_d its class group, $h_d = |C_d|$. The Weyl sum is

$$W(f_1, f_2; d) = \frac{1}{h_d} \sum_{t \in C_d} \Phi_1(t) \overline{\Phi_2(t)} \qquad (\Phi_i(t) = \varphi_i(t.u_i)).$$

Main estimate

Under GRH, we have $W(f_1, f_2; d) \ll_\epsilon (\log |d|)^{-1/4+\epsilon}$.

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View as inner product on class group C_d. Plancherel formula gives

$$W(f_1, f_2; d) = \sum_{\chi \in \widehat{C_d}} W_1(f_1, \chi; d)\overline{W_2(f_2, \chi; d)}.$$

Heuristic (under GRH):

- roughly $\approx |d|^{1/2}$ terms in the sum,
- each term is roughly $W_1(f_1, \chi; d)\overline{W_2(f_2, \chi; d)} \approx |d|^{-1/2}$

Might *hope* for square-root cancellation: $W(f_1, f_2; d) \ll |d|^{-1/4}$.

Crazy first step: void all cancellation!

$$|W(f_1, f_2; d)| \leq \sum_{\chi \in \widehat{C_d}} |W_1(f_1, \chi; d) W_2(f_2, \chi; d)|.$$

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Assume $W_1(f_1, \chi; d)\overline{W_2(f_2, \chi; d)} \neq 0$. Twisted Waldspurger gives

$$|W_i(f_i, \chi; d)|^2 \doteq |d|^{-1/2} \frac{L(1/2, \pi_i \times \chi)}{L(1, \eta_d)^2 L(1, \mathrm{Ad}\, \pi_i)}.$$

Get (using class number formula)

$$|W(f_1, f_2; d)| \leq \mathcal{L}_d(1) S(d),$$

where $\mathcal{L}_d(1) = L(1, \eta_d)^{-2} L(1, \mathrm{Ad}\, \pi_1)^{-1/2} L(1, \mathrm{Ad}\, \pi_2)^{-1/2}$ and

$$S(d) = \frac{1}{h_d} \sum_{\chi \in \widehat{C_d}} L(1/2, \pi_1 \times \chi)^{1/2} L(1/2, \pi_2 \times \chi)^{1/2}.$$

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Note that $\pi_1 \neq \pi_2$ since $\boldsymbol{G}_1 \not\simeq \boldsymbol{G}_2$. Show $S(d) \ll_\epsilon (\log |d|)^{-1/4+\epsilon}$.

Pointwise GRH fails (as it must)

Cauchy-Schwartz reduces this to bounding

$$\frac{1}{h_d} \sum_{\chi \in \widehat{C_d}} L(1/2, \pi \times \chi) \leq \max_{\chi \in \widehat{C_d}} L(1/2, \pi \times \chi).$$

Clearly subconvexity is not going to do the job!

Under GRH (and Ramanujan), we have the general bound:

$$L(1/2, \pi) \ll \exp(A \log C(\pi) / \log \log C(\pi))$$

Moreover (Soundararajan), there exist $d \in [X, 2X]$ such that

$$L(1/2, \eta_d) \gg \exp(c \sqrt{\log X} / \log \log X).$$

One can expect similar lower bounds on $L(1/2, \pi \times \chi)$ for $\chi \in \widehat{C_d}$.

Structurally similar situation: unipotent coefficients

QUE for arithmetic eigenfunctions (AQUE) on the modular surface.

- even weight (holomorphic): Holowinsky (2009):

$$\frac{1}{T}\sum_{n\sim T}|\lambda_f(n)\lambda_f(n+1)| \ll (\log T)^{-\delta}$$

- $1/2$-integral weight (Maass): Lester–Radziwiłł (2019) on GRH:

$$\frac{1}{T}\sum_{d\sim T} L(1/2, f\times\eta_d)^{1/2} L(1/2, f\times\eta_{d+1})^{1/2} \ll (\log T)^{-\delta}.$$

On average, these unipotent coefficients are of size $\approx (\log n)^{-\delta}$, *independently* on small shifts.

$$\mathrm{PGL}_2 \times \mathrm{PGL}_2$$

$$\boldsymbol{N} \times \boldsymbol{N}$$

$$\Delta \boldsymbol{N}$$

Proof of main estimate

Let $h = h_d$ and

$$L_1(\chi) = L(1/2, \pi_1 \times \chi)^{1/2} \quad \text{and} \quad L_2(\chi) = L(1/2, \pi_2 \times \chi)^{1/2}.$$

View $\log L_1(\chi)$ as independent Gaussian random variables in χ.

Put $L(\chi) = L_2(\chi) L_2(\chi)$.

Let μ and σ^2 be the expectation and variance of $\log L(\chi)$:

$$\mu = \frac{\mu_1 + \mu_2}{2} \qquad \text{and} \qquad \sigma^2_{\text{naive}} = \frac{\sigma_1^2 + \sigma_2^2}{4}.$$

Can calculate each μ_i and σ_i^2 under GRH: for small x

$$\log L(1/2, \pi_i \times \chi) \lesssim \sum_{p \le x} \frac{\lambda_{\pi_i}(p) a_\chi(p)}{p^{1/2}} + \frac{1}{2} \sum_{\substack{p^2 \le x \\ \eta_d(p) = 1}} \frac{\lambda_{\pi_i}(p^2) a_\chi(p^2)}{p} + \mu_i.$$

The important feature is that $\exp\left(\mu + \frac{\sigma^2_{\text{naive}}}{2}\right) \asymp (\log |d|)^{-1/4}$.

Proof (continued)

By partial summation we obtain

$$S(d) = \frac{1}{h} \sum_{\chi} L(\chi) = \frac{1}{h} \int_{\mathbb{R}} e^{V} \#\{\chi : \log L(\chi) > V\} dV$$
$$= e^{\mu} \int_{\mathbb{R}} e^{V} N(V) dV,$$

where

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Now, for any $k \geq 0$, we have

$$N(V) \leq V^{-2k} M_{2k}(V),$$

where

$$M_{2k}(V) = \frac{1}{h}\sum_{\chi}\big(\log L(\chi) - \mu\big)^{2k}.$$

Proof (end)

By orthogonality of characters, we show, say for $k \gg \log\log|d|$,

$$M_{2k}(V) \ll \frac{(2k)!}{k!}\Big(\frac{\sigma^2}{2}\Big)^k,$$

where

$$\sigma^2 = \sigma^2_{\mathrm{naive}} + \log L(1, \pi_1 \times \pi_2 \times \theta_d)^{1/2}.$$

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upon choosing $k = V^2/(2\sigma^2)$. Get

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This approach dates back to Soundarajan (2009), on moments of the Riemann zeta function.

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Thank You!