Time quasi-periodic gravity water waves

Massimiliano Berti
SISSA, Trieste

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Joint work with P. Baldi, E. Haus, R. Montalto

1-d gravity Water Waves: Euler equations for an irrotational, incompressible fluid in $S_\eta(t) := \{-h < y < \eta(t,x)\}$ under gravity

$$
\begin{cases}
\partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + g\eta = 0 & \text{at } y = \eta(t,x) \\
\Delta \Phi = 0 & \text{in } -h < y < \eta(t,x) \\
\partial_y \Phi = 0 & \text{at } y = -h \\
\partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } \quad y = \eta(t,x)
\end{cases}
$$

$$
u = \nabla \Phi = \text{velocity field}, \ \ \mathrm{rot}\, u = 0 \ (\text{irrotational}),
$$
$$
\mathrm{div}\, u = \Delta \Phi = 0 \ (\text{uncompressible})
$$
$$
g = \text{gravity},
$$

Unknowns:

free surface $y = \eta(t,x)$ and the velocity potential $\Phi(t,x,y)$

Zakharov formulation '68

Infinite dimensional Hamiltonian system:

$$\partial_t u = J \nabla_u H(u) , \quad u := \begin{pmatrix} \eta \\ \psi \end{pmatrix} , \quad J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} ,$$

canonical Darboux coordinates:

$\eta(x)$ and $\psi(x) = \Phi(x, \eta(x))$ trace of velocity potential at $y = \eta(x)$

(η, ψ) uniquely determines Φ in the whole $\{-h < y < \eta(x)\}$ solving the elliptic problem:

$$\Delta\Phi = 0 \quad \text{in } \{-h < y < \eta(x)\}, \quad \Phi|_{y=\eta} = \psi, \quad \partial_y \Phi = 0 \text{ at } y = -h$$

Hamiltonian: total energy on $S_\eta = \mathbb{T} \times \{-h < y < \eta(x)\}$

$$H := \frac{1}{2} \int_{S_\eta} |\nabla \Phi|^2 dxdy + \int_{S_\eta} gy\, dxdy$$

kinetic energy + potential energy

Hamiltonian expressed in terms of (η, ψ)

$$H(\eta, \psi) = \tfrac{1}{2} \int_{\mathbb{T}} \psi(x)\, G(\eta)[\psi](x)\, dx + \int_{\mathbb{T}} g \tfrac{\eta^2}{2}\, dx$$

Dirichlet–Neumann operator, Craig-Sulem '93, non-local operator

$$G(\eta)[\psi](x) := \sqrt{1+\eta_x^2}\, \partial_n \Phi|_{y=\eta(x)} = (\Phi_y - \eta_x \Phi_x)(x, \eta(x))$$

Zakharov-Craig-Sulem formulation

$$\begin{cases} \partial_t \eta = G(\eta)\psi = \nabla_\psi^{L^2} H(\eta,\psi) \\ \partial_t \psi = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{\left(G(\eta)\psi + \eta_x \psi_x\right)^2}{2(1+\eta_x^2)} = -\nabla_\eta^{L^2} H(\eta,\psi) \end{cases}$$

Dirichlet–Neumann operator

$$G(\eta)\psi(x) = G(\eta,h)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n \Phi|_{y=\eta(x)}$$

1. $G(\eta)$ is linear in ψ,
2. self-adjoint with respect to $L^2(\mathbb{T}_x)$
3. $G(\eta) \geq 0$, $G(1) = 0$
4. $\eta \mapsto G(\eta)$ nonlinear, smooth
5. $G(\eta)$ is pseudo-differential, $G(\eta) = D_x \tanh(hD_x) + OPS^{-\infty}$

Alazard, Burq, Craig, Delort, Lannes, Metivier, Zuily, ...

Symmetries

Reversibility

$$H(\eta, -\psi) = H(\eta, \psi)$$

Involution

$$H \circ S = H, \quad S : (\eta, \psi) \to (\eta, -\psi), \quad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad S^2 = \mathrm{Id},$$

Reversible vector field $X_H = J\nabla H$

$$X_H \circ S = -S \circ X_H \quad \Longleftrightarrow \quad \Phi_H^t \circ S = S \circ \Phi_H^{-t}$$

Equivariance under the $\mathbb{Z}/(2\mathbb{Z})$-action of the group $\{\mathrm{Id}, S\}$

Reversible solutions

$$u(t) = Su(-t) \quad \Longleftrightarrow \quad \eta(t) = \eta(-t)\,, \ \psi(t) = -\psi(-t)$$

Standing Waves

Invariant subspace: functions even in x

$$\eta(-x) = \eta(x)\,, \quad \psi(-x) = \psi(x)$$

Thus the velocity potential

$$\Phi(-x, y) = \Phi(x, y)$$

and, using also 2π periodicity,

$$\Phi_x(0, y) = \Phi_x(\pi, y) = 0$$

$\implies$ no flux of fluid outside the walls $\{x = 0\}$ and $\{x = \pi\}$.

Prime integral: mass

$$\int_{\mathbb{T}} \eta(x)dx$$

$$\eta \in H_0^s(\mathbb{T}) := \{\eta \in H^s(\mathbb{T}) \, : \, \int_{\mathbb{T}} \eta(x)dx = 0\}$$

The variable ψ is defined modulo constants: only the velocity field $\nabla_{x,y}\Phi$ has physical meaning.

$$\psi \in \dot{H}^s(\mathbb{T}) = H^s(\mathbb{T})/\sim$$

$$u(x) \sim v(x) \quad \Longleftrightarrow \quad u(x) - v(x) = c$$

Linear water waves theory

Linearized system at $(\eta, \psi) = (0, 0)$,

$$\begin{cases} \partial_t \eta = G(0)\psi \\ \partial_t \psi = -g\eta \end{cases}$$

Dirichlet-Neumann operator at the flat surface $\eta = 0$ is

$$G(0) = D \tanh(hD)\,, \quad D = \frac{\partial_x}{\mathrm{i}} = \mathrm{Op}(\xi)_{\xi \in \mathbb{R}}$$

Fourier multiplier notation: given $m : \mathbb{Z} \to \mathbb{C}$

$$m(D)h = \mathrm{Op}(m)h = \textstyle\sum_{j \in \mathbb{Z}} m(j) h_j e^{\mathrm{i}jx}\,, \quad h(x) = \sum_{j \in \mathbb{Z}} h_j e^{\mathrm{i}jx}$$

Linear water waves system

$$\partial_t \begin{bmatrix} \eta \\ \psi \end{bmatrix} = \begin{bmatrix} 0 & G(0) \\ -g & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \psi \end{bmatrix}$$

Complex variables

$$u = \Lambda(D)\eta + \mathrm{i}\Lambda^{-1}(D)\psi\,, \quad \Lambda(D) = \left(\frac{g}{D\tanh(hD)}\right)^{1/4}$$

Linear WW

$$u_t + i\omega(D)u = 0, \quad \omega(D) = \sqrt{g}\sqrt{D\tanh(hD)}$$

∞-decoupled Harmonic oscillators

$$u(t,x) = \sum_{j\in\mathbb{Z}} e^{-\mathrm{i}\omega(j)t} u_j(0) e^{\mathrm{i}jx}$$

Linear frequencies of oscillations

$$\omega(j) = \sqrt{g}\sqrt{j\tanh(hj)}\,, \quad j \in \mathbb{Z}\,,$$

All solutions are periodic, quasi-periodic, almost periodic in time according to the irrationality properties of $(\omega_j(h))_{j\in\mathbb{Z}}$

Question

Are there periodic, quasi-periodic, almost periodic solutions of the full nonlinear gravity water waves equations?

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Nonlinear water waves

Major difficulties:

1) Nonlinear gravity Water Waves are a fully-linear system

$$u_t + \mathrm{i}\omega(D)u = N(u, \bar{u})$$

$N =$ quadratic nonlinearity with derivatives of higher order with respect to the unperturbed linear ones: $N = N(|D|u)$

Singular perturbation of the linear vector field $\mathrm{i}\omega(D)u$

2) Linear dispersion relation: sublinear at infinity

$$\omega(j) = \sqrt{j \tanh(hj)} \sim \sqrt{|j|}, \quad j \to \infty,$$

3) Verify non-resonance conditions

The parameter h moves just exponentially the frequencies

Remarks

1. Boundary conditions: $x \in \mathbb{T}$ periodic $\implies$ *NO dispersive* effects of the linear PDE as for $x \in \mathbb{R}^2$, $x \in \mathbb{R}$ and data dacaying at infinity:
   Global well-posedness: S. Wu, Germain-Masmoudi-Shatah, Ionescu-Pusateri, Alazard-Delort, Ifrim-Tataru, Alazard-Burq-Zuily, ...
2. Pure gravity waves. With capillarity

$$\omega(D) = \sqrt{D \tanh(hD)(g + \kappa D^2)} \sim \sqrt{\kappa}|D|^{\frac{3}{2}}$$

M.B. - J.M. Delort, '17: almost global existence result

Quasi-periodic solution with n frequencies of $u_t = J\nabla H(u)$

Definition

$u(t,x) = U(\omega t, x)$ where $U(\varphi, x) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$,

$\omega \in \mathbb{R}^n$(= *frequency vector*) is irrational $\omega \cdot k \neq 0\,,\ \forall k \in \mathbb{Z}^n \setminus \{0\}$

$\Longrightarrow$ the linear flow $\{\omega t\}_{t\in\mathbb{R}}$ is DENSE on $\mathbb{T}^n$

The torus-manifold

$$\mathbb{T}^n \ni \varphi \mapsto U(\varphi, x) \in \text{phase space}$$

is *invariant* under the flow Φ_H^t of the PDE

$$\Phi_H^t \circ U = U \circ \Psi_\omega^t$$

$$\Psi_\omega^t : \mathbb{T}^n \ni \varphi \to \varphi + \omega t \in \mathbb{T}^n$$

Linear reversible solutions

$$\eta(t,x) = \sum_{j\in\mathbb{S}} \sqrt{\xi_j}\cos(\omega_j t)\cos(jx)\,, \qquad \xi_j > 0$$
$$\psi(t,x) = -\sum_{j\in\mathbb{S}} \sqrt{\xi_j}\omega_j^{-1}\sin(\omega_j t)\cos(jx)$$

where $\mathbb{S} = \{\bar{\jmath}_1, \ldots, \bar{\jmath}_n\} \subset \mathbb{Z}$ is a finite set (tangential sites)

Finite dimensional invariant tori for the linearized water-waves eq.

$$\eta(\varphi,x) = \sum_{j\in\mathbb{S}} \sqrt{\xi_j}\cos(\varphi_j)\cos(jx)\,,$$
$$\psi(\varphi,x) = -\sum_{j\in\mathbb{S}} \sqrt{\xi_j}\omega_j^{-1}(h)\sin(\varphi_j)\cos(jx)$$

ANGLES: $\varphi = (\varphi_j)_{j\in\mathbb{S}} \in \mathbb{T}^n$, FREQUENCIES: $\vec{\omega}(h) = (\omega_j(h))_{j\in\mathbb{S}}$

- *Do they persist in the nonlinear Water Waves?*

Periodic solutions: $n = 1$

- **Plotnikov-Toland**: '01
  Gravity Water Waves with Finite depth
- **Iooss-Plotnikov-Toland** '04, **Iooss-Plotnikov** '05-'09
  Gravity Water Waves with Infinite depth
  Completely resonant, infinite dimensional bifurcation equation
- **Alazard-Baldi** '15,
  Capillary-gravity water waves with infinite depth

  No information about their linear stability
- QUESTION: WHAT ABOUT QUASI-PERIODIC SOLUTIONS?

KAM theory:

1. Semilinear perturbations of NLS and NLW : '80-'90 Kuksin, Wayne, Craig, Bourgain, Pöschel, etc
2. Semilinear nonlinearities with derivatives: '97 Kuksin (KdV), '97 Bourgain (DNLW), '02 Kappeler-Pöschel (KdV), '11 Liu-Yuan (DNLS), '12 Berti-Biasco-Procesi (DNLW)
3. Higher space dimension: Bourgain, Eliasson, Kuksin, Grebert, Berti, Bolle, Procesi, Wang,
4. Quasi-linear/fully non-linear perturbations:
    1. Baldi-Berti-Montalto '13 - '15, KdV
       key issue: use of pseudo-differential calculus
    2. Feola-Procesi, NLS
    3. Berti-Montalto '16, Gravity-Capillary water waves

KAM for gravity Water Waves: main result

Look for small amplitude quasi-periodic solutions

$$(\eta(t,x),\psi(t,x)) = (\eta(\tilde\omega t,x),\psi(\tilde\omega t,x))$$

of

Gravity Water Waves equations

$$\begin{cases} \partial_t \eta = G(\eta)\psi \\ \partial_t \psi = -\eta - \dfrac{\psi_x^2}{2} + \dfrac{\left(G(\eta)\psi + \eta_x \psi_x\right)^2}{2(1+\eta_x^2)} \end{cases}$$

with frequencies $\tilde\omega = (\tilde\omega_j)_{j\in\mathbb{S}}$ (to be found) close to linear frequencies $(\omega_j(h))_{j\in\mathbb{S}}$, $\omega_j(h) := \sqrt{j\tanh(hj)}$

Theorem (Baldi, Berti, Haus, Montalto, 2017)

For every choice of the tangential sites $\mathbb{S} \subset \mathbb{N} \setminus \{0\}$, *there exists* $\bar{s} > \frac{|\mathbb{S}|+1}{2}$, $\varepsilon_0 \in (0,1)$ *such that: for all* $\xi_j \in (0, \varepsilon_0^2)$, $j \in \mathbb{S}$, $\exists$ *a Cantor like set* $\mathcal{G}_\xi \subset [h_1, h_2]$ *with asymptotically full measure as* $\xi \to 0$, *i.e.* $\lim_{\xi \to 0} |\mathcal{G}_\xi| = h_2 - h_1$, *such that, for any depth* $h \in \mathcal{G}_\xi$, *the* GRAVITY WATER WAVES EQUATION *has a reversible, quasi-periodic standing wave solution* $(\eta, \psi) \in H^{\bar{s}}$ *of the form*

$$\eta(t,x) = \sum_{j \in \mathbb{S}} \sqrt{\xi_j} \cos(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|})$$

$$\psi(t,x) = -\sum_{j \in \mathbb{S}} \sqrt{\xi_j} \omega_j^{-1} \sin(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{|\xi|})$$

with frequencies $\tilde{\omega}_j$ *satisfying* $\tilde{\omega}_j - \omega_j(h) \to 0$ *as* $\xi \to 0$.
The solutions are **linearly stable**.

Remarks

1. The restriction to $\mathcal{G}_\xi$ is not technical. Otherwise there could be "Chaos", "Arnold Diffusion", "growth of Sobolev norms" ... The system is expected to be not integrable **Craig-Workfolk** '94, **Zakharov** '95: the $5-th$ order (formal) Birkhoff normal form system is not integrable for $h=+\infty$
2. There are no global existence results for solutions of the water waves equations with periodic boundary conditions: **the previous theorem selects initial conditions which give rise to smooth solutions defined for all times**
3. There is not an almost global existence result as for capillary-gravity water waves of Berti-Delort. The depth h moves just exponentially the frequency:

$$h \to \sqrt{j \tanh(hj)}\,, \quad \tanh(hj) = 1 + O(e^{-hj})$$

This is enough for quasi-periodic solutions!

Linear stability -reducibility

There exist coordinates (around the torus)

$$(\phi, y, v) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times (H^s_x \cap L^2_{\mathbb{S}^c})$$

in which the quasi-periodic solution reads $t \mapsto (\tilde{\omega} t, 0, 0)$ and the linearized equation reads

$$\begin{cases} \dot{\phi} = by \\ \dot{y} = 0 \\ v_t = \mathrm{i} D_\infty v \,, \quad v = \sum_{j \notin \mathbb{S}} v_j e^{\mathrm{i} jx} \,, \; D_\infty := \mathrm{Op}(\mu_j) \,, \; \mu_j \in \mathbb{R} \end{cases}$$

$$y(t) = y_0, v_j(t) = v_j(0) e^{\mathrm{i}\mu_j t} \Longrightarrow \|v(t)\|_{H^s_x} = \|v(0)\|_{H^s_x} \; : \text{ stability}$$

$$0, \{\mathrm{i}\mu_j\}_{j \in \mathbb{S}^c} = \textbf{Floquet exponents}$$

1. Sharp **asymptotic expansion** of the **Floquet exponents**

$$\mu_j = m_{\frac{1}{2}} (j \tanh(hj))^{\frac{1}{2}} + r_j(h)$$

where $m_{\frac{1}{2}} \in \mathbb{R}$ is a constant satisfying

$$|m_{\frac{1}{2}} - 1| + \sup_{j \in \mathbb{S}^c} |j|^{\frac{1}{2}} |r_j| = O(|\xi|^a), \ a > 0,$$

2. The change of variables $\Phi(\varphi)$ satisfies **tame estimates** in Sobolev spaces:

$$\|\Phi h\|_s, \|\Phi^{-1} h\|_s \leq \|h\|_s + \|u\|_{s+\sigma} \|h\|_{s_0}, \ \forall s \geq s_0$$

Small-divisors problem. Look for $u(\varphi, x) := (\eta(\varphi, x), \psi(\varphi, x))$ zero of

$$\mathcal{F}(\omega, h, u) := \begin{pmatrix} \omega\cdot\partial_\varphi \eta - G(\eta)\psi \\ \omega\cdot\partial_\varphi \psi + \eta - \dfrac{\psi_x^2}{2} - \dfrac{\left(G(\eta)\psi + \eta_x\psi_x\right)^2}{2(1+\eta_x^2)} \end{pmatrix}$$

Small amplitude solutions:

$$\mathcal{F}(\omega, h, 0) = 0 \quad D_{\eta,\psi}\mathcal{F}(\omega, h, 0) = \begin{pmatrix} \omega\cdot\partial_\varphi & -D\tanh(hD) \\ 1 & \omega\cdot\partial_\varphi \end{pmatrix}$$

In Fourier basis

$$D_{\eta,\psi}\mathcal{F}(\omega, h, 0) = \mathrm{diag}_{\ell\in\mathbb{Z}^\nu, j\in\mathbb{Z}} \begin{pmatrix} \mathrm{i}\omega\cdot\ell & -j\tanh(hj) \\ 1 & \mathrm{i}\omega\cdot\ell \end{pmatrix}$$

- QUESTION: is $D_u\mathcal{F}(\omega, h, 0)$ invertible?

$$\det\begin{pmatrix} \mathrm{i}\omega\cdot\ell & -j\tanh(hj) \\ 1 & \mathrm{i}\omega\cdot\ell \end{pmatrix} = -(\omega\cdot\ell)^2 + j\tanh(hj)$$

Non-resonance condition:

$$\left| -(\omega\cdot\ell)^2 + j\tanh(hj)\right| \geq \frac{\gamma}{\langle \ell\rangle^{\tau}}, \forall(\ell, j)\in\mathbb{Z}^n\times\mathbb{Z}, (\ell, j)\neq 0, \ \tau > 0$$

$\Longrightarrow D_u\mathcal{F}(\omega, h, 0)$ is invertible, but the inverse is **unbounded**:

$$D_u\mathcal{F}(\omega, h, 0)^{-1} : H^s \to H^{s-\tau}, \ \tau := \text{"}\textit{LOSS OF DERIVATIVES}\text{"}$$

In addition the nonlinearity of the Water Waves equations contains derivatives of higher order!

Nash-Moser Implicit Function Theorem

Newton tangent method for zeros of $\mathcal{F}(u) = 0$ + "smoothing":

$$u_{n+1} := u_n - S_n (D_u \mathcal{F})^{-1}(u_n) \mathcal{F}(u_n)$$

where S_n = regularizing operators

- Difficulty: invert $(D_u\mathcal{F})(u)$ in a whole neighborhood of 0 with *tame* estimates of the inverse

$$\|(D_u\mathcal{F})(u)^{-1} h\|_s \leq \|h\|_{s+\sigma} + \|u\|_{s+\sigma_1} \|h\|_{s_0}\,, \quad \forall s \geq s_0$$

Linearized operator at $u(\varphi, x) = (\eta, \psi)(\varphi, x)$

$$(D_u\mathcal{F})(u) = \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x \circ V + G(\eta) \circ B & -G(\eta) \\ (1 + BV_x) + B \circ G(\eta) \circ B & V \circ \partial_x - B \circ G(\eta) \end{pmatrix}$$

where $(V, B) = \nabla_{x,y} \Phi$ are smooth functions and

$$G(\eta) = D_x \tanh(hD_x) + R_\infty(\eta)\,, \quad R_\infty \in OPS^{-\infty}$$

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Linearized operator at $u(\varphi, x) = (\eta, \psi)(\varphi, x)$

$$(D_u \mathcal{F})(u) =$$
$$\omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x \circ V + G(\eta) \circ B & -G(\eta) \\ (1 + B V_x) + B \circ G(\eta) \circ B & V \circ \partial_x - B \circ G(\eta) \end{pmatrix}$$

where $(V, B) = \nabla_{x,y} \Phi$ are smooth functions and

$$G(\eta) = D_x \tanh(h D_x) + R_\infty(\eta) , \quad R_\infty \in OPS^{-\infty}$$

Ideas of Proof:

1. Nash-Moser implicit function theorem formulated like a "Théorème de conjugaison hypothétique" à la Herman
2. **Degenerate KAM theory:**
   non-resonance conditions and measure estimates
3. **Diagonalization of linearized PDE in normal subspace**
   1. **Step 1.** **Reduction of linearized PDE up to very smoothing operators**
      Pseudo-differential theory in **original physical coordinates**
   2. **Step 2.** **KAM reducibility scheme**.
      Very weak second order Melnikov conditions

Degenerate KAM theory

A big issue in KAM theory: fullfill non-resonance conditions

- Choose parameters
- Non-degeneracy conditions:
  1. Kolmogorov
  2. Arnold-Piartly
  3. ...
  4. Rüssmann (Fejoz-Herman for Celestial Mechanics)

  weakean as much as possible the non-degeneracy conditions

  Use h **(**$=$ **depth) as a parameter** (i.e. spacial period $2\pi\lambda$)

Linear frequencies $\omega_j(h) = \sqrt{j \tanh(hj)}$

Difficulties

1. $\omega_j(h) = \sqrt{j \tanh(hj)} \sim \sqrt{j}$
2. $\omega_j(h) = \sqrt{j \tanh(hj)} = \sqrt{j} + O(e^{-hj})$
   (for gravity-capillary $\omega_j(\kappa) = \sqrt{(1+\kappa j^2) j \tanh(hj)} \sim j^{3/2}$)

Degenerate KAM theory for PDEs, Bambusi-Berti-Magistrelli,

Analyticity: $h \mapsto \omega_j(h)$ + *Non-degeneracy property* + simple eigenvalues $\Longrightarrow$

Lemma (Transversality)

$\exists k_0 \in \mathbb{N}$, $\rho > 0$ such that: $\forall(\ell, j, j') \neq (0, j, j)$, $h \in [h_1, h_2]$,

$$\max_{k \leq k_0} |\partial_h^k \{\vec{\omega}(h) \cdot \ell + \omega_j(h) - \omega_{j'}(h)\}| \geq \rho \langle \ell \rangle$$

$\Longrightarrow$

2-order Melnikov conditions which lose derivatives also in space

$$|\vec{\omega}(h) \cdot \ell + \omega_j(h) - \omega_{j'}(h)| \geq \gamma \frac{\langle \ell \rangle^{-\tau}}{j^{\mathrm{d}} j'^{\mathrm{d}}}, \forall(\ell, j, j') \neq (0, j, j), j, j' \in \mathbb{N}^+ \setminus \mathbb{S}$$

Will be compensated by a highly regularizing remainder

Non degeneracy property: torsion Lemma

$\forall N$, $\forall 1 \leq j_1 < \ldots < j_N$, the curve

$$[h_1, h_2] \ni h \mapsto \Big(\omega_{j_1}(h), \ldots, \omega_{j_N}(h)\Big) \in \mathbb{R}^N$$

is not contained in any hyperplane of $\mathbb{R}^N$

PROOF:

Analyticity + Generalized Van der Monde determinant

Reduction of linearized operator in normal subspace

Linearized equation $\partial_t h = J\partial_u \nabla H(u(\omega t, x))h$

$$\mathcal{L}_\omega := \omega \cdot \partial_\varphi + \Pi_{\mathbb{S}}^{\perp} \begin{pmatrix} \partial_x V + G(\eta)B & -G(\eta) \\ (1 + BV_x) + BG(\eta)B & V\partial_x - BG(\eta) \end{pmatrix} \Pi_{\mathbb{S}}^{\perp}$$

GOAL:

Conjugate $\mathcal{L}_\omega$ to a diagonal operator (Fourier multiplier) constant in φ:

$$\Phi^{-1} \circ \mathcal{L}_\omega \circ \Phi = \omega \cdot \partial_\varphi + \mathrm{diag}\{\mathrm{i}\mu_j(\varepsilon, \omega, h)\}_{j \in \mathbb{S}^c}$$

where

$$\mu_j = m_{\frac{1}{2}} (j \tanh(hj))^{\frac{1}{2}} + r_j(h), \quad \sup_j j^{1/2} r_j = O(\varepsilon)$$

1. **"Diagonalization of $\mathcal{L}_\omega$ up to smoothing operators"**

$$\mathcal{L}_1 := \Phi^{-1}\mathcal{L}_\omega\Phi = \omega\cdot\partial_\varphi + \mathrm{i} m_{\frac{1}{2}} D\tanh(hD) + \mathrm{i} r(D) + R_0$$

- $m_{\frac{1}{2}} \in \mathbb{R}$, $r(\xi) \in \mathbb{R}$, $r(\xi) \in S^{-\frac{1}{2}}$ **constant in** (φ, x),
- $R_0(\varphi, x)$ is regularizing $O(\partial_x^{-M})$ and satisfies tame estimates

2. **"Reduction of the size of R_0"**

$$\mathcal{L}_n := \Phi_n^{-1}\mathcal{L}_1\Phi_n = \omega\cdot\partial_\varphi + \mathrm{i} m_{\frac{1}{2}} D\tanh(hD) + \mathrm{i} r^{(n)}(D) + R_n$$

- *KAM quadratic scheme:* $R_n = O(\varepsilon^{2^n})$, $r^{(n)}(\xi) \sim r(\xi) \in \mathbb{R}$,

second order Melnikov conditions with loss of derivatives: $M \gg 1$

Transformation laws

Linear system, quasi-periodic in time,

$$u_t + A(\omega t)u = 0$$

Under a transformation of the phase space, quasi-periodic in time,

$$u = \Phi(\omega t)[v], \quad \mathbb{T}^\nu \ni \varphi \mapsto \Phi(\varphi) \in \mathcal{L}(H^s)$$

becomes

New linear system

$$v_t + B(\omega t)v = 0$$
$$B(\omega t) = \Phi^{-1}(\omega t)(\omega \cdot \partial_\varphi \Phi)(\omega t) + \Phi(\omega t)^{-1}A(\omega t)\Phi(\omega t)$$

Our GOAL: $B = \mathrm{Op}(\mu_j)$ Fourier multiplier with μ_j constants!
Choose $A(\omega t)$: it will be the composition of several changes of variables, of different nature

Step 1) Alinhac good-unknown

$$\mathcal{L}_\omega := \omega\cdot\partial_\varphi + \begin{pmatrix} \partial_x \circ V + G(\eta)\circ B & -G(\eta) \\ (1+B\circ V_x) + B\circ G(\eta)\circ B & V\circ\partial_x - B\circ G(\eta) \end{pmatrix}$$

Alinhac good unknown

$$Z(\varphi) = \begin{pmatrix} 1 & 0 \\ B(\varphi,x) & 1 \end{pmatrix}, Z(\varphi)^{-1} = \begin{pmatrix} 1 & 0 \\ -B(\varphi,x) & 1 \end{pmatrix}, \left[\begin{smallmatrix} \widehat{\eta} \\ \widehat{\omega} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widehat{\eta} \\ \widehat{\psi} - B\widehat{\eta} \end{smallmatrix}\right]$$

New system

$$\mathcal{L}_0 = Z^{-1}\mathcal{L}_\omega Z = \omega\cdot\partial_\varphi + \begin{pmatrix} \partial_x\circ V & -G(\eta) \\ a & V\circ\partial_x \end{pmatrix}, \quad a = a(\varphi,x)$$

Remark: Z is reversibility preserving

Step 2) Singular perturbation term: $V\partial_x$

$$\omega \cdot \partial_\varphi + \begin{pmatrix} V(\varphi, x)\partial_x & 0 \\ 0 & V(\varphi, x)\partial_x \end{pmatrix} + \begin{pmatrix} V_x & -G(\eta) \\ a & 0 \end{pmatrix}$$

Goal

Conjugate the vector field $\omega \cdot \partial_\varphi + V(\varphi, x)\partial_x$ to the constant coefficient one $\omega \cdot \partial_\varphi$ on the torus $\mathbb{T}^\nu_\varphi \times \mathbb{T}_x$ for $V(\varphi, x)$ small.

1. Perturbative KAM argument to *straighten* the vector field. Difficulty: $\omega \cdot \partial_\varphi$ is resonant on $\mathbb{T}^\nu_\varphi \times \mathbb{T}_x \implies$ use reversibility
2. Quasi-periodic transport PDE: *all* solutions of carachteristic system $\dot{x} = V(\omega t, x)$ are quasi-periodic with frequency $\omega \implies$ $\exists$ foliation of quasi-periodic solutions
3. Remark: for periodic solutions, it is not a small divisor problem

New system, after symmetrization + complex variables, reads

Quasi-linear perturbation

$$\omega \cdot \partial_\varphi + \mathrm{i} a_1(\varphi, x)(D \tanh(hD))^{\frac{1}{2}} + a_2(\varphi, x) + \dots$$

Conjugating with flow of

$$\partial_\tau u = \mathrm{i}\beta(\varphi, x)|D|^{\frac{1}{2}} u \,,$$

we get

$$\omega \cdot \partial_\varphi + \mathrm{i}(\omega \cdot \partial_\varphi \beta(\varphi, x) + a_1(\varphi, x))(D \tanh(hD))^{\frac{1}{2}} + \dots$$

Choice of β

$$\omega \cdot \partial_\varphi + \mathrm{i}\langle a_1 \rangle(x)(D \tanh(hD))^{\frac{1}{2}} + a_3(\varphi, x) + \dots$$

Conjugating with a diffeomorphism

$$x \mapsto x + \alpha(x) \quad \Longrightarrow$$

we get

$$\omega \cdot \partial_\varphi + \mathrm{i}\langle a_1\rangle(x)(1 + \alpha_x(x))(D\tanh(hD))^{\frac{1}{2}} + \dots$$

Choice of α

$$\omega \cdot \partial_\varphi + \mathrm{i}m_{\frac{1}{2}}(D\tanh(hD))^{\frac{1}{2}} + a_4(\varphi, x) + \dots$$

finally, applying a sequence of quasi-periodically time dependent pseudo-differential transformations one obtains

$$\Phi^{-1} \circ \mathcal{L}_\omega \circ \Phi = \omega \cdot \partial_\varphi + \mathrm{i}m_{\frac{1}{2}} D\tanh(hD) + \mathrm{i}r_{-\frac{1}{2}}(D) + R_0$$

$R_0 = O(\partial_x^{-M})$ variable coefficients (φ, x)

KAM diagonalization

Reduce the size of the perturbation $R_0(\varphi, x)$

Second order Melnikov non-resonance conditions

$$|\vec{\omega}(h) \cdot \ell + \Omega_j(h) - \Omega_{j'}(h)| \geq \gamma \frac{\langle \ell \rangle^{-\tau}}{j^d j'^d}, \forall (\ell, j, j') \neq (0, j, j)$$

DIFFICULTY: Implement a KAM reducibility scheme with very weak Melnikov non-resonance conditions.
Compensated by the high order of regularization that we performed

$$R_0 = O(\partial_x^{-M}), \qquad M \gg d$$

Conclusion:

$$\mathcal{L}_\infty := \Phi_\infty^{-1} \mathcal{L}_\omega \Phi_\infty = \omega \cdot \partial_\varphi + \mathrm{i} m_{\frac{1}{2}} D \tanh(hD) + \mathrm{i} r^{(\infty)}(D)$$