I). ~~Intro~~ Intro: $G/\mathbb{Q}$ conn. red. gp. $\rightsquigarrow$ X symm. space for $G(\mathbb{R})$ | Princeton talk ①

$K \subset G(\mathbb{A}_f)$ neat c.o. subgp $\rightsquigarrow$ $X_K = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$

Examples: 1). $G = SL_2/\mathbb{Q} \rightsquigarrow X = \mathcal{H}^2$, X_K = modular curves

general case: Shimura varieties, eg. $G = GSp_{2n}$, unitary gp.

2). $G = \mathrm{Res}_{F/\mathbb{Q}} SL_2$, F imaginary quadratic $\rightsquigarrow X = \mathcal{H}^3$, X_K = arithmetic hyperbolic 3-manifolds

general case: $\mathrm{Res}_{F/\mathbb{Q}} GL_n$, F number field.

$E/\mathbb{Q}_p$ fin, $\mathcal{O} \subset E$ integers, $\varpi \in \mathcal{O}$ uniformiser, $\mathbb{F} = \mathcal{O}/\varpi$.

Let σ_λ be irred alg rep'n of G over E of highest wt $\lambda \in (\mathbb{Z}^n)^{\mathrm{Hom}_{\mathbb{Q}}(F,\bar{\mathbb{Q}})}$

$\cup$

σ_λ° K_p-stable $\mathcal{O}$-lattice $\rightsquigarrow$ $\mathcal{V}_\lambda$ local system of $\mathcal{O}$-modules on X_K.

Interested in $H^i_{(c)}(X_K, \mathcal{V}_\lambda) \circlearrowleft \mathbb{T}$ Hecke algebra $(*)$

- $H^i_{(c)}(X_K, \mathcal{V}_\lambda)[1/p]$ can be described in terms of aut reps of $GL_n(\mathbb{A}_F)$
- but $(*)$ can contain torsion!

Calegari-Geraghty (2011) extended Taylor-Wiles method to this setting.

- Conditional on detailed understanding of $(*)$
- Unconditional implementation: Let F be a CM field, E/F a non-CM

(ACCGHLNSTT) 2018 elliptic curve. Then E is pot automorphic & satisfies Sato-Tate

- Also coherent version $\rightsquigarrow$ Boxer-Calegari-Gee-Pilloni

II). The CG method relies on 3 conjectures.

1. Existence of Galois reps: Let $\mathbb{T}(K,\lambda) = \mathrm{Im}\left(\mathbb{T} \to \mathrm{End}_{\mathcal{O}}\left(H^*_{(c)}(X_K, \mathcal{V}_\lambda)\right)\right)$

Let $\mathfrak{m} \subset \mathbb{T}(K,\lambda)$ be a max'l ideal

$\Downarrow$ conj

$\bar{\rho}_{\mathfrak{m}} : G_F = \mathrm{Gal}(\bar{F}/F) \to GL_n(\bar{\mathbb{F}})$ cts, semi-simple, st. Satake params of $\mathfrak{m}$ match Frobenius eigenvalues of $\bar{\rho}_{\mathfrak{m}}$ at all but finitely many good primes.

Assume $\bar{\rho}_{\mathfrak{m}}$ irreducible.

$\Downarrow$ conj

$\rho_{\mathfrak{m}} : \mathrm{Gal}(\bar{F}/F) \to GL_n(\mathbb{T}(K,\lambda)^\wedge_{\mathfrak{m}})$ lift of $\bar{\rho}_{\mathfrak{m}}$ cts, semisimple, sat. compatibility at good primes.

If F is a CM field, this is OK by Scholze up to $I \subset \mathbb{T}(K,\lambda)^\wedge_{\mathfrak{m}}$ nilpotent ideal.

2. Local-global compatibility. Let $v \mid \ell$ be a prime of F

Expect (K_v, λ_v) to determine ramification of $\rho_{\mathfrak{m}}|_{G_{F_v}}$

- $\ell \neq p$ Varma, Helm's integral Bernstein center
- $\ell = p$ tricky: construction of $\rho_{\mathfrak{m}}$ loses track of level at p, HT wts, & difficult to formulate integrally

Gee-Newton: use Kisin's crystalline def rings $\boxed{\text{keep track of}}$

$$\begin{tikzcd} R^{\square}_{\bar{\rho}_{\mathfrak{m}}|G_{F_v}} \arrow[r] \arrow[dr] & R^{univ}_{\bar{\rho}_{\mathfrak{m}}} \arrow[r] & \mathbb{T}(K,\lambda)^\wedge_{\mathfrak{m}}/I \\ & R^{\square}_{\bar{\rho}_{\mathfrak{m}}|G_{F_v}}(\lambda_v) \arrow[ur, dashed] & \end{tikzcd}$$

crystalline, HT wts det by λ.

3). Vanishing of cohomology : $X = G(\mathbb{R}) / K_\infty A_\infty$

define invariants

$$\begin{cases} l_0 := \mathrm{rk}\, G(\mathbb{R}) - \mathrm{rk}\, K_\infty - \mathrm{rk}\, A_\infty \\ q_0 := \frac{1}{2}(\dim X - l_0). \end{cases} \rightsquigarrow [q_0, q_0 + l_0]$$

symmetric about middle degree

Examples: $SL_2/\mathbb{Q} \rightsquigarrow l_0 = 0,\ q_0 = 1.$

$\mathrm{Res}_{F/\mathbb{Q}} SL_2$, F im. quad $\rightsquigarrow l_0 = 1, q_0 = 1.$

Assume $\mathfrak{m}$ nonEis. Then expect $H^i_{c}(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}} = 0$ unless $i \in [q_0, q_0 + l_0]$.

- OK in char 0 (Franke, Borel-Wallach)
- progress for Shimura varieties $\rightsquigarrow l_0 = 0$, recently
- low-dim'l examples.
- Khare-Thorne: reduce to char 0 statement, OK.

III. LGC:

Thm 1 (C-Scholze, 2019): Let $\tilde{X}_{\tilde{K}}$ be a unitary Shimura variety attached to split $\tilde{G}/\mathbb{Q}$. Let $\tilde{\mathfrak{m}} \subset \mathbb{T}(\tilde{K}, \tilde{\lambda})$ be "generic".

Then i). $H^i_c(\tilde{X}_{\tilde{K}}, \mathcal{V}_{\tilde{\lambda}}/\varpi)_{\tilde{\mathfrak{m}}} = 0 \Rightarrow i \leq d = \dim_{\mathbb{C}} \tilde{X}_{\tilde{K}}$

ii). $H^i(\tilde{X}_{\tilde{K}}, \mathcal{V}_{\tilde{\lambda}}/\varpi)_{\tilde{\mathfrak{m}}} = 0 \Rightarrow i \geq d.$

Cor: Have $\mathbb{T}(\tilde{K}, \tilde{\lambda})$-equivariant diagram:

$$(\heartsuit)\quad H^d(\tilde{X}_{\tilde{K}}, \mathcal{V}_{\tilde{\lambda}})_{\tilde{\mathfrak{m}}}[1/p] \hookleftarrow H^d(\tilde{X}_{\tilde{K}}, \mathcal{V}_{\tilde{\lambda}})_{\tilde{\mathfrak{m}}} \twoheadrightarrow \underset{\in [0,d-1]}{H^{?}}(\partial \tilde{X}^{BS}_{\tilde{K}}, \mathcal{V}_{\tilde{\lambda}})_{\tilde{\mathfrak{m}}}$$

Given $(K, \lambda, i, \mathfrak{m})$, can find $(\tilde{K}, \tilde{\lambda}, \tilde{\mathfrak{m}})$ st.

$(*)$ contributes to $(\heartsuit)$ "degree-shifting".

Thm 2 (C-Newton, in progress). Assume that:

1). $F = F^+ \cdot F_0$, p split in F_0, $\bar{v}$ prime of F^+ above p.
2). $K_v = GL_n(\mathcal{O}_{F,v})$ $\forall\, v \mid \bar{v}$ in F
3). $[F^+_{\bar{v}} : \mathbb{Q}_p] \leq \frac{1}{2}[F^+ : \mathbb{Q}]$
4). $\bar{\rho}_m$ irred & generic.

Then $(**)$ factors as desired, up to possibly enlarging I.

Key idea: Let $P \subset \widetilde{G}$ be Siegel parabolic w/ Levi G.
Consider version of (♡) for P-ordinary completed coh at $\bar{v}$
Combine P-ordinary degree-shifting at $\bar{v}$ + strengthened FL degree shifting at $\bar{v}' \neq \bar{v}$.

$\rightsquigarrow$ congruence between $\rho_m \oplus \rho_m^{\vee,c}$ to $\rho_{\tilde{m}}$ s.t. $\rho_{\tilde{m}}|_{G_{F^+_{\bar{v}}}} = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ $n \times n$ blocks. $\square$

conclude by thm of T. Liu char mod p^n pts of $R_{\bar{\rho}_m|G_{F_{\bar{v}}}}(\lambda)$

Rmks: 1). method should generalise to pst ss case :)
2). cond 3) excludes IQF's. :(
3). should generalise ALT's of ten authors paper. :)