SQG equation on the sphere

Analysis seminar, IAS

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Active scalars are a wide class of *transport equations* where the velocity is determined from the transported quantity in a certain way. In this case a temperature θ evolves following

$$\begin{cases} \partial_t \theta(x,t) + u(x,t)\cdot\nabla\theta(x,t) = 0, \\ \nabla\cdot u(x,t) = 0 \\ u(x,t) = (-R_2\theta, R_1\theta) \end{cases}$$

where R_j denotes a Riesz potential

$$R_j(\theta)(x) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{y_j}{|y|^3}\theta(x-y)dy$$

Motivation

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2. It is related to the 3D Euler equation.

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A glimpse of history: the origins

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D. Córdoba. Provided a proof dismissing such an scenario, Annals of Math. 1998.

In this case we consider

$$\begin{cases} \theta_t + u \cdot \nabla\theta + \kappa\Lambda^{\alpha}\theta = 0 \\ u = \nabla^{\perp}\Lambda^{-1}\theta \\ \theta(x,0) = \theta_0(x), \kappa > 0 \end{cases}$$

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The subcritical case, $\alpha > 1$, is well understood (Constantin-Wu, 1998). Global regularity for the critical case $\alpha = 1$ has been quite more challenging due to the possible balance between opposite strengths of the non-linear and the dissipative terms.

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1. Existence (local or global).
2. Uniqueness.
3. Finite time singularities (even in the compressible case).

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Global well-posedness results for critical dissipation

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In fact all their proofs work in the n-dimensional tori or euclidean spaces. Our result deals only with the two dimensional sphere, namely:

Alonso-Orán, Córdoba, M. ($\mathbb{S}^2$): Global well-posedness of critical SQG on the sphere, 2018.

Let (M, g) be a compact orientable surface and g be a Riemannian metric the SQG in this case takes the form

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + u \cdot \nabla_g \theta + \Lambda_g \theta = 0, \\ u = \mathcal{R}_g^\perp \theta = \nabla_g^\perp \Lambda_g^{-1} \theta \end{cases}$$

where $\Lambda_g = (-\Delta_g)^{\frac{1}{2}}$ and $-\Delta_g$ is the Laplace-Beltrami operator.

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Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)

Let $\theta_0 \in C^\infty(\mathbb{S}^2)$ be the initial datum, then the solution remains smooth for any time $t > 0$.

Sketch

Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)

Given an initial datum $\theta_0 \in L^2(\mathbb{S}^2)$ any weak solution becomes instantaneously continuous for any time $t > 0$.

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Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)

There is global well-posedness in $H^s(\mathbb{S}^2)$ for any $s > 3$. In fact, any solution with such initial datum becomes smooth instantaneously.

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$$\Lambda^\alpha f(x) = c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+\alpha}} dy,$$
$$\Lambda^\alpha f(x) = c_{n,\alpha} P.V. \sum_{\nu \in \mathbb{Z}^n} \int_{\mathbb{T}^n} \frac{f(x)-f(y)}{|x-y-\nu|^{n+\alpha}} dy$$

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for any $0<\alpha<2$. As an easy consequence of these one can get

Theorem (Córdoba-Córdoba inequality)

For any $\alpha \in (0,2)$ and f smooth enough, the following pointwise inequality holds

$$f(x)\Lambda^\alpha f(x) \geq \frac{1}{2}\Lambda^\alpha (f^2)(x).$$

Following the trajectories

$$\frac{dx}{dt} = u(x,t)$$

one gets

$$\frac{d}{dt}(\theta(x(t),t)) = \theta_t + \nabla\theta \cdot \frac{dx}{dt} = 0$$

deducing that $\|\theta(\cdot,t)\|_{L^p}$ remains constant under the evolution $(1 \leq p \leq \infty)$.

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deducing that $\|\theta(\cdot,t)\|_{L^p}$ remains constant under the evolution $(1 \leq p \leq \infty)$. It is easy to check, using the Córdoba-Córdoba inequality, that the L^p norms do not increase.

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The following pointwise inequality holds

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PROOF: To begin with we propose the following Dirichlet problems in the domain

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{in } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = f^{2m} & \text{in } \partial\Omega \end{cases}$$

Then $w = u^{2m} - v$ satisfies

$$\begin{cases} \Delta w = 2m(2m-1)|\nabla u|^2 u^{2m-2} & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega \end{cases}$$

The (spectral) fractional Laplace-Beltrami (definition)

The Laplace-Beltrami operator, Δ_g, in some local coordinates of the surface $(n = 2)$ takes the form

$$\frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{2} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \right)$$

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$$-\Delta_g Y_\ell = \lambda_\ell^2 Y_\ell$$

where $\lambda_0 = 0$ and the eigenvalue increases to infinity as $\ell \in \mathbb{N}$ increases.

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In the case of a general compact manifold (e.g. a sphere) the above was not available.

Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)

$$\Lambda_g^\alpha f(x) = c_{n,\alpha} P.V. \int_M \frac{f(x)-f(y)}{d(x,y)^{n+\alpha}} (\chi u_0 + k_N)(x,y)\, dy + E$$

where $k_N(x,y) = O(d(x,y))$ *is certain smooth function,* χ *is a diagonal cutoff and the error gains derivatives*

$$E = O(\|f\|_{H^{-N}(M)}).$$

.

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- Constantin-Vicol improvement with smoothing error.
- Fractional Sobolev embedding theorem for compact manifolds (cf. Aubin).

Recall from our previous sketch

Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)

Given an initial datum $\theta_0 \in L^2(\mathbb{S}^2)$ any weak solution becomes instantaneously continuous for any time $t > 0$.

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Difficulties:

1. No dilations available.
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3. The argument does not work for dimensions greater than two.

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Lemma (Caccioppoli's (energy) inequality)

Let $u \geq 0$, $\Delta u \geq 0$ *and* $\varphi \in C_0^\infty(B_2)$ *then*

$$\int_{B_2} |\nabla(\varphi u)|^2(x)dx \leq C_\varphi \int_{B_2} u^2.$$

De Giorgi's technique 101

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Using Sobolev's embedding and this inequality one can prove

$$E_k \leq C2^{2k} E_{k-1}^{1+1/n}$$

where

$$E_k = \int_{B_1} (\varphi_k u_k)^2 dx$$

where $u_k = (u - (1 - 2^{-k}))_+$ and φ_k is a cut off function on $B_{1+2^{-k}}$.

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where $u_k = (u - (1 - 2^{-k}))_+$ and φ_k is a cut off function on $B_{1+2^{-k}}$. Then $E_k \to 0$ if E_0 is small enough.

The previous idea can be used, this time

$$E_k = \sup_{t \geq T_k} \int_M \theta_k^2 dx + 2 \int_{T_k}^{\infty} \int_M |\Lambda^{1/2} \theta_k|^2 dx dt$$

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to show

$$E_k \leq C \frac{2^{k(1+2/n)+1}}{t_0 C^{2/n}} E_{k-1}^{1+1/n}.$$

Lemma (Local energy inequality, I)

Let θ_k satisfy

$$\partial_t \theta_k + u \cdot \nabla_g \theta_k \leq -\Lambda \theta_k$$

and denote $I(z_0) = [0, z_0]$. Let the function $\eta\theta_k^*(x,t,z)$ be vanishing in $M \times [0,\infty) \setminus B_g(h) \times I(z_0)$. Then if u satisfies

$$\sup_{t \in (s,t)} \int_{B_g(h)} |u(x,t)|^{2n} dvol_g(x) \leq C h^n$$

and $s \leq t$.

Lemma (Local energy inequality, II)

Then the following holds

$$\int_s^t \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z}(\eta\theta_k^*)(x,t,z)|^2 dxdzdt + \int_{B_g(h)} (\eta\theta_k)^2(x,t)dx$$
$$\leq C\left\{\int_{B_g(h)} (\eta\theta_k)^2(x,s)dx + h\int_s^t \int_{B_g(h)} |\nabla_x \eta\theta_k|^2 dxdt\right.$$
$$+ \int_s^t \int_{I(z_0)} \int_{B_g(h)} |\nabla_{x,z}\eta\theta_k^*|^2 dvol_g(x)dzdt$$
$$\left.+ \int_s^t \int_{B_g(h)} (\eta\theta_k)^2(x,t)dvol_g(x)dt\right\}.$$

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- Fractional Sobolev embedding to get global $L^\infty(M)$ control out of L^2 norm.
- Nice barrier functions satisfying certain properties (recall no scaling!).
- Very tricky induction argument to get oscillation decay using the above (for small energy).
- Isoperimetric inequality on the sphere (cf. De Giorgi's to get rid of the small energy assumption).
- Rotations to get the logarithmic modulus of continuity.

Non linear maximum principle

Lemma

Let f be a smooth function on the sphere $\mathbb{S}^2$ and $0 < \alpha < 2$. Then provided that $|\nabla_g f(x)| \geq C\|f\|_\infty$ we have the pointwise bound

$$\nabla_g f(x)\cdot\nabla_g \Lambda^\alpha f(x) \geq \frac{1}{2}\Lambda^\alpha(|\nabla_g f|^2)(x)+\frac{1}{4}D(x)+\frac{|\nabla_g f(x)|^{2+\alpha}}{C\|f\|_\infty^\alpha}+O$$

where D denotes some functional (defined in the proof), $O = O(\|\nabla_g f\|_\infty^2)$ is an error term and the constant C depends on α but is independent of x.

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where D denotes some functional (defined in the proof), $O = O(\|\nabla_g f\|_\infty^2)$ is an error term and the constant C depends on α but is independent of x.

Careful: notice the non commutativity of some operators (despite ΨDO). We need stereographic projection, integral representation with smoothing error.

Closing the argument (I)

Let $L = (\partial_t + u \cdot \nabla_g + \Lambda_g)$.

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$$\frac{1}{2}L(|\nabla_g \theta|^2)(x) + \frac{|\nabla_g \theta(x)|^3}{c\|\theta\|_\infty} \leq C|\nabla_g \theta(x)|^2 + O(\|\nabla_g \theta\|_\infty^2)$$

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holds for any $t > 0$. Evaluating formally at $\bar{x}$, a point that reaches the maximum of $|\nabla_g \theta|(x)$, one gets

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Heuristically this prevents indefinite growth for the L^∞ norm of the gradient.This heuristic can be replaced by a honest argument.

From the above it follows that

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in fact, it will be bounded by some absolute constant $C > 0$.This control is all one needs to prove the theorem thanks to the usual energy estimates.

Theorem (D. Alonso-Orán, A. Córdoba, A. D. M., 2018)

There is global well-posedness in $H^s(\mathbb{S}^2)$ for any $s > 3$. In fact, any solution with such initial datum becomes smooth instantaneously.

Thank you!