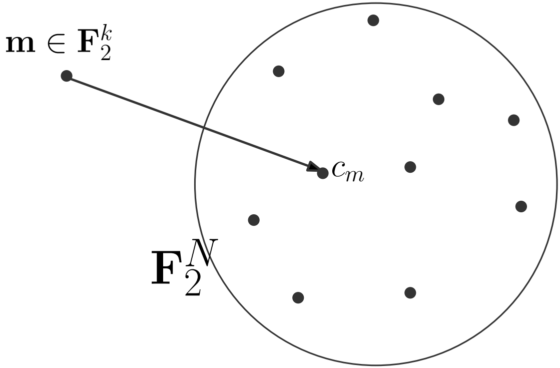
Higher-order Fourier Analysis over Finite Fields,
and Applications

Pooya Hatami

Coding Theory:

Task: Reliably transmit a message through an unreliable channel.



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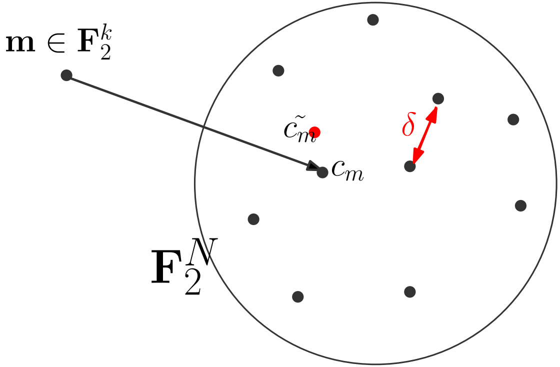

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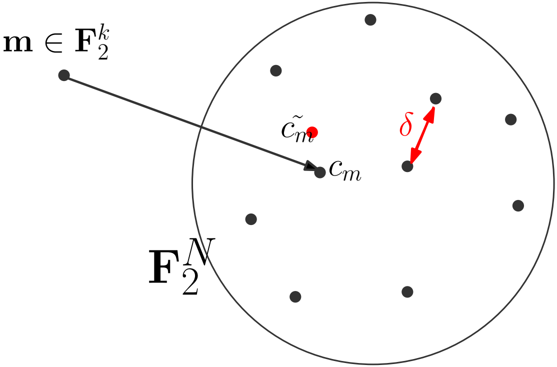

Good code:

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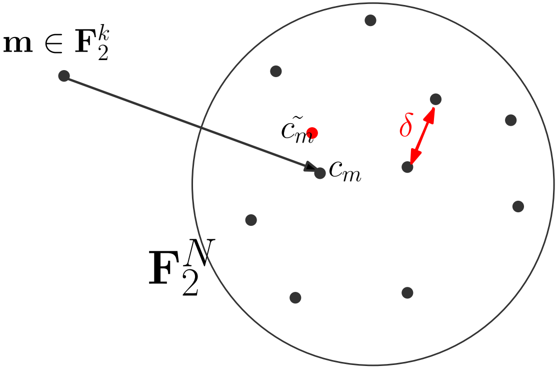
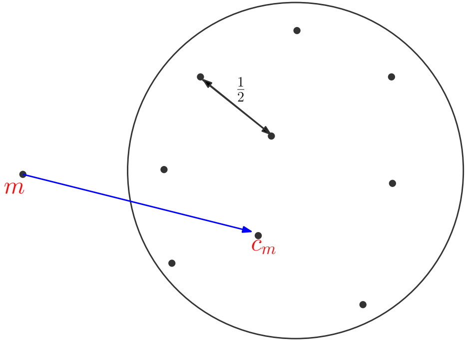


Good code:

(i) Large minimum distance δ while having large rate $\frac{k}{N}$.

(ii) Efficiently testable and decodable.

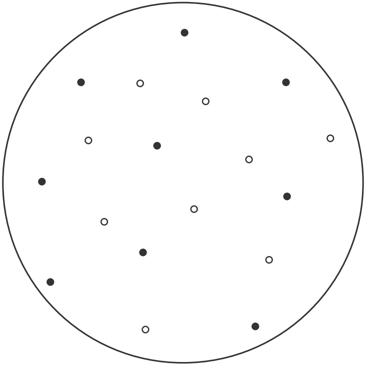
Hadamard Codes:



$$m \in \mathbb{F}_2^k \longrightarrow c_m \in \mathbb{F}_2^{2^k}$$

c_m: evaluation vector of $m_1 x_1 + m_2 x_2 + \cdots + m_k x_k \in \mathbb{F}_2[x_1, \ldots, x_k]$

Reed Muller codes:



Degree $\leq d$ polynomials in $\mathbf{F}_2[x_1, ..., x_k]$

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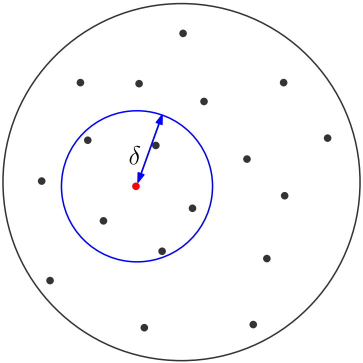

Problem 1.

How many degree $\leq d$ polynomials in $\mathbb{F}_2[x_1, ..., x_n]$ are there in $B_\delta(f)$?

Polynomial Decompositions:

Degree 4 polynomial $P \in \mathbb{F}_2[x_1, ..., x_n]$.

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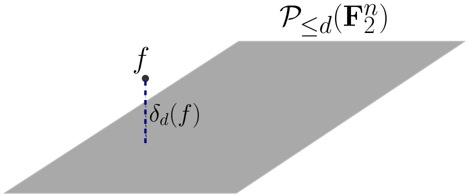
Algebraic Property Testing:

Is $f : \mathbb{F}_2^n \to \mathbb{F}_2$ a degree d polynomial?



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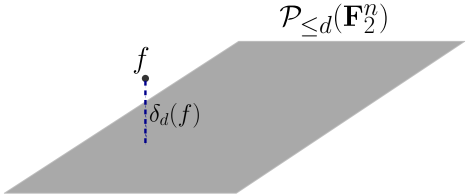

[AKKLR'05] Query f only on constant number of inputs.

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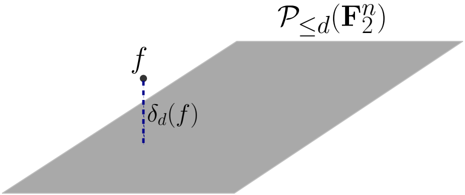

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Problem 3.
Which "algebraic" properties are testable?

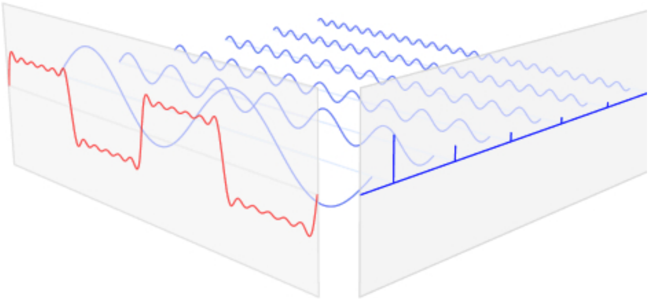
Higher-order Fourier analysis over finite fields,

which is an extension of Fourier analysis.

[Bergelson, Green, Kaufman, Gowers, Lovett, Meshulam, Samorodnitsky, Tao, Viola, Wolf . . .]

Fourier Analysis over $\mathbb{F}_p^n$

Study $f : \mathbb{F}_p^n \to \mathbb{R}$ by looking at how it correlates with linear phases.



$$f(x) = \sum_{\sigma \in \mathbb{F}_p^n} \widehat{f}(\sigma) \cdot \omega^{\sum \sigma_i x_i}$$

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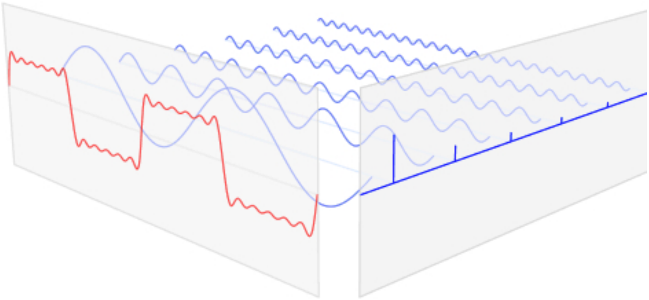

$$f(x) = \sum_{\sigma : |\widehat{f}(\sigma)| \geq \epsilon} \widehat{f}(\sigma) \cdot \omega^{\sum \sigma_i x_i} + f_{psd}$$

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Study $f : \mathbb{F}_p^n \to \mathbb{R}$ by looking at how it correlates with higher degree polynomial phases, $\omega^{P(x_1,\ldots,x_n)}$.

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- ◇ $f = \sum_{i=1}^{C} \lambda_i \omega^{P_i(x)} + f_{psd}$?

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[Bergelson, Green, Tao, Ziegler] establish such decomposition theorems via inverse theorems for certain norms called Gowers norms.

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For any collection $\mathcal{G}$ of functions $g : X \to \mathbb{D}$ the following holds.

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◇ Need to understand the joint distribution of a collection of degree d polynomials.

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Problem: $P_1, \ldots, P_C \in \mathcal{P}_{\leq d}(\mathbb{F}_p^n)$. $X, Y \in \mathbb{F}_p^n$ uniform. Characterize the distribution of

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[HHL '15 (general case), BFHHL'13 (affine linear forms)]

[KL'08 and GT'09]: Distribution of $(P_1(X), ..., P_C(X))$.

Problem 1. [KLP '10, BL '15]

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Problem 4. Is there a constant query tester that given $f : \mathbb{F}_2^n \to \mathbb{F}_2$ distinguishes between the following?

- f is $\geq \epsilon$-correlated to some cubic, or
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where $0 < \delta(\epsilon) \leq \epsilon$.

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Theorem. [BHT]

There is a $\mathrm{poly}(n)$-time deterministic algorithm that given a polynomial P, and $\Gamma : \mathbb{F}_p^\ell \to \mathbb{F}_p$, and $d_1, ..., d_\ell \geq 1$, either

- outputs $P_1, ..., P_r$ of degrees $d_1, ..., d_\ell$, s.t. $P = \Gamma(P_1, ..., P_d)$, or
- correctly outputs **NOT POSSIBLE**.

Proof illustration: Find P_1, P_2 of degree $\leq d-1$ such that

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- ▷ Recurse on $P|_{x_j=0}$.
  - If **NOT POSSIBLE**, then output **NOT POSSIBLE**.
  - Otherwise we find P_1', P_2' such that $P|_{x_j=0} = P_1' P_2'$.

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