NODAL DOMAINS FOR MAASS FORMS

PETER SARNAK

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$\mathbb{H}$ UPPER HALF PLANE

$\Gamma = SL_2(\mathbb{Z})$

$X = \Gamma \backslash \mathbb{H}$ MODULAR SURFACE

THE EVERYWHERE UNRAMIFIED AUTOMORPHIC CUSP FORMS FOR $GL_2/\mathbb{Q}$ ARE THE EVEN HECKE/MAASS FORMS ON X:

$$\Delta\phi + \lambda\phi = 0 \ , \ \phi \in L^2_{cusp}(X)$$

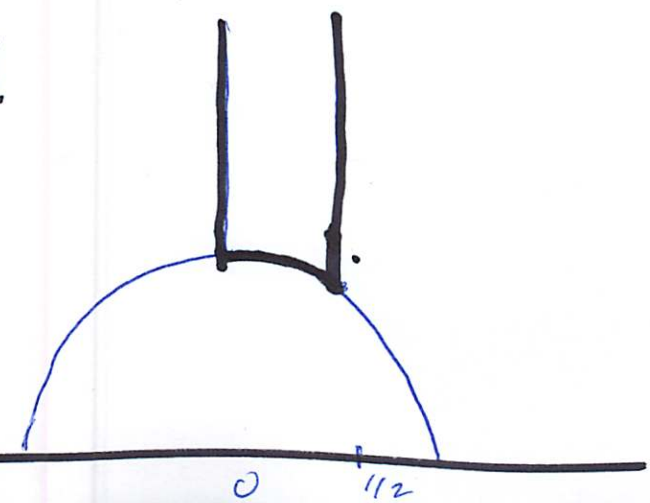
$$\phi(\sigma z) = \phi(z) \ , \ \sigma : X \to X$$

THE INVOLUTION INDUCED BY

$$z \to -\bar{z}$$

$\mathrm{FIX}(\sigma) = \delta =$

$\partial_n \phi |_\delta = 0$ / NEUMANN COND.



OUR INTEREST IS IN THE NODAL DOMAINS OF ϕ.

$$Z(\phi) = \{ z \in X : \phi(z) = 0 \}, \text{ NODAL LINE}$$

$$X \setminus Z(\phi) = \bigsqcup_{j=1}^{N} \Omega_j \quad , \Omega_j \text{ CONNECTED}$$

COMPONENTS : NODAL DOMAINS.

HOW MANY ARE THERE?

THE NUMBER MEETING A LARGE BUT FIXED COMPACT $K \subset X$ SATISFIES

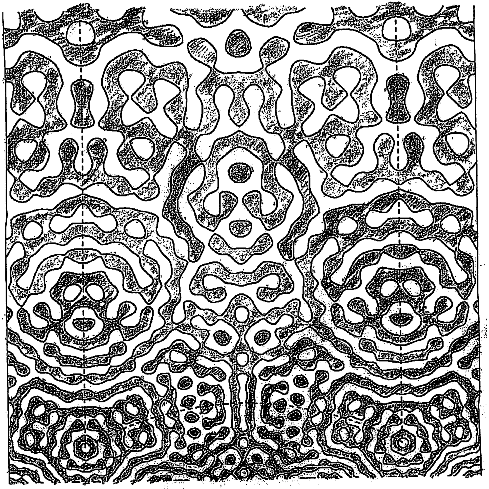
$$N_K(\phi_\lambda) \ll \lambda \quad \text{(COURANT)}.$$

[FOR COMPACT MANIFOLDS IN ANY DIMENSION

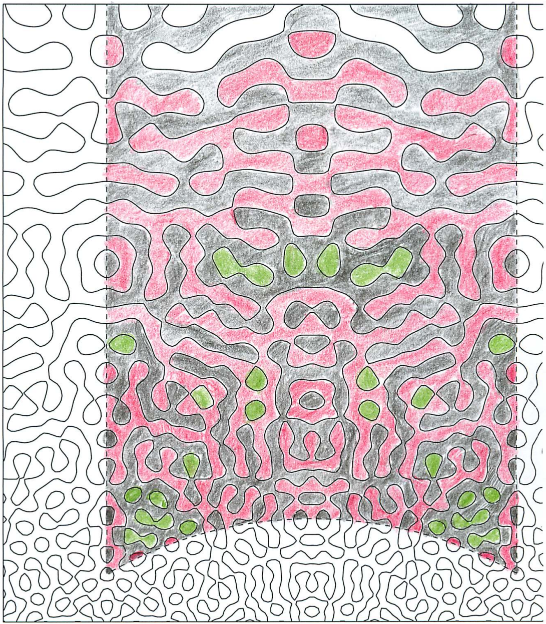
$$N(\phi_n) \leq n$$

WHERE n DENOTES THE n-TH EIGENFUNCTION]

Below a picture of the zero set $\phi = 0$ of such a "Maass form" for $\mathrm{SL}_2(\mathbb{Z})$, $\lambda = \frac{1}{4} + t^2$, $t = 125.34\ldots$ (Hejhal–Rackner).
Is the zero set behaving randomly? How many components does it have?

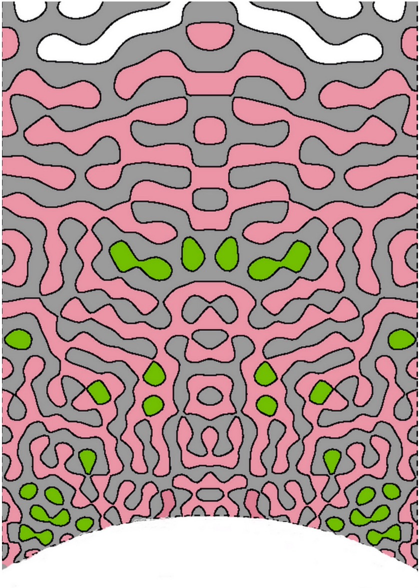


58 nodal domains in A

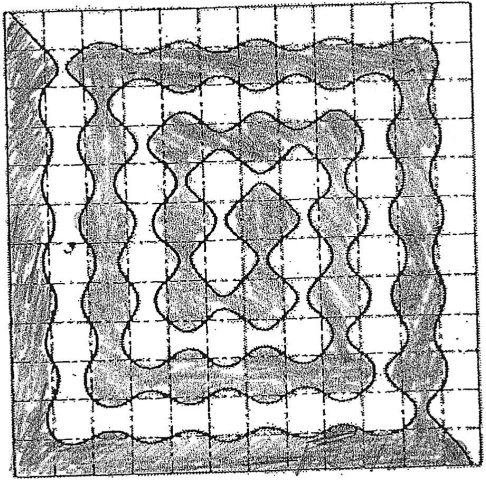


Hejhal–Rackner nodal lines for $\lambda = 1/4 + R^2$, $R = 125.313840$

Nodal portrait



The trouble is giving a lower bound for $N(\phi)$. In general it need not grow!



Nodal domain of an eigenfunction on the square, $N(\phi) = 2$.
From Courant–Hilbert; Vol I. Thesis A. Stern, Gottingen, 1925.

WHAT SHOULD WE EXPECT ?

M. BERRY: THE EIGENSTATES OF THE QUANTIZATION OF A CLASSICALLY CHAOTIC HAMILTONIAN IN THE SEMI-CLASSICAL LIMIT BEHAVE LIKE RANDOM MONOCHROMATIC WAVES.

- THE CLASSICAL GEODESIC MOTION ON

$$T_1^*(X) \cong \Gamma \backslash SL_2(\mathbb{R})$$

IS WELL KNOWN TO BE CHAOTIC.

RANDOM MONOCHROMATIC WAVES:

S^2 (SIMILAR DEFN IN ANY DIMENSION)

H_k: THE $2k+1$ DIMENSIONAL SPACE OF OF SPHERICAL HARMONICS OF DEGREE k. THESE YIELD THE EIGENFUNCTIONS OF Δ_{S^2} WITH EIGENVALUE $\lambda_k = \frac{k(k+1)}{2}$.

ϕ_j, $j = -k, \ldots, k$ O.N.B OF H_k

$$\psi_\lambda(z) = \sum_{j=-k}^{k} a_j \, \phi_j(z)$$

WITH a_j I.I.D. $N(0,1)$ VARIABLES.

ψ_λ IS A RANDOM WAVE WITH FREQ. λ.

WHAT DO ITS NODAL DOMAINS LOOK LIKE?

<u>THEOREM 1</u> (NAZAROV-SODIN 2008)

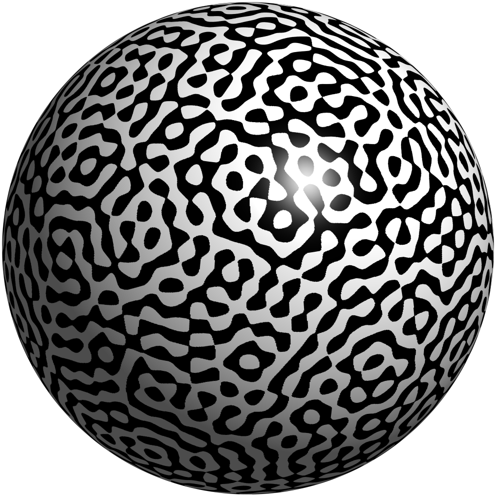
AS $\lambda \to \infty$ THE RANDOM ψ_λ HAS

$$N(\psi_\lambda) \sim c\lambda$$

FOR SOME UNIVERSAL $c > 0$.

FOR $m \geq 1$ LET $N_m(\psi_\lambda)$ BE THE NUMBER OF NODAL DOMAINS OF ψ WHICH HAVE CONNECTIVITY m.

Nodal portrait: Random spherical harmonic ($\alpha = 1$)



random spherical harmonic of degree $= 80$. (A. Barnett)

THEOREM 2: (CANZANI-WIGMAN-S 2017)
THERE IS A PROBABILITY MEASURE μ ON $\mathbb{N}$ WITH $\mu(m)>0$ FOR ALL m S.T. FOR RANDOM ψ

$$N_m(\psi_\lambda) \sim \mu(m)\, N(\psi_\lambda) \text{ AS } \lambda \to \infty.$$

NUMERICAL MONTE CARLO EXPERIMENTS (NASTASESCU, KONRAD, BARNETT, JIN, ...) YIELD VALUES ;

$c \approx 1/20$ (RIGOROUS BOUND $c \geq 10^{-70}$ DE COURCY-IRELAND)

$\mu(1) = 0.906$, $\mu(2) = 0.055$,
$\mu(3) = 0.010$, $\mu(4) = 0.006$, ...

CHECKING THESE NUMBERS AGAINST THE NODAL DOMAINS OF MAASS FORMS ON $SL_2(\mathbb{Z})$ GIVES GOOD AGREEMENT.

- SO WE EXPECT AN INDIVIDUAL WEYL TYPE LAW FOR THE NODAL COUNT.

REMARK: THE THEOREMS FOR THE NODAL SETS AND DOMAINS OF RANDOM MONOCHROMATIC WAVES CAN BE CARRIED OUT IN ANY DIMENSION YIELDING A UNIVERSAL DISTRIBUTION FOR THE TOPOLOGIES OF $Z(\psi)$ (C-W-S). HOWEVER THE CORRESPONDING CONSTANTS IN DIMENSION 3 AND HIGHER APPEAR TO BE ABSURDLY SMALL. WE APPEAR TO BE IN A SUPERCRITICAL REGIME (NUMERICAL EXPERIMENTS - BARNETT) SO THAT FOR PRACTICAL PURPOSES THERE ESSENTIALLY TWO NODAL DOMAINS!

WHAT CAN ONE PROVE FOR MAASS FORMS AND HOW TO PRODUCE NODAL DOMAINS?

<u>THEOREM 3</u> (GHOSH-REZNIKOV-S 2014):

$\Gamma = SL_2(\mathbb{Z})$, ϕ AS ABOVE, K A FIXED LARGE COMPACT IN X, ASSUME THE LINDELOFF HYPOTHESIS FOR $L(s, \phi_\lambda)$ THEN

$$N(\phi_\lambda) \gg \lambda^{1/24} .$$

MORE GENERALLY LET X BE A CONGRUENCE ARITHMETIC SURFACE (I.E. A CONGRUENCE COVER OF ~~E~~ THE MODULAR SURFACE OR A SHIMURA "CURVE")

ASSUME THAT X CARRIES A REFLECTION SYMMETRY $\sigma : X \to X$.

$\delta = \mathrm{FIX}(\sigma) \subset X$ IS A FINITE UNION OF CLOSED GEODESICS.

WE ASSUME THAT ALL ~~MAASS~~ MAASS EIGENFUNCTION ϕ ARE ALSO EIGENFUNCTIONS OF σ.

THEOREM 4 (G-R-S 2017):

ASSUMING THE LINDELOFF HYPOTHESIS FOR GL_2 AUTOMORHIC L-FUNCTIONS (OVER ANY F) WE HAVE THAT

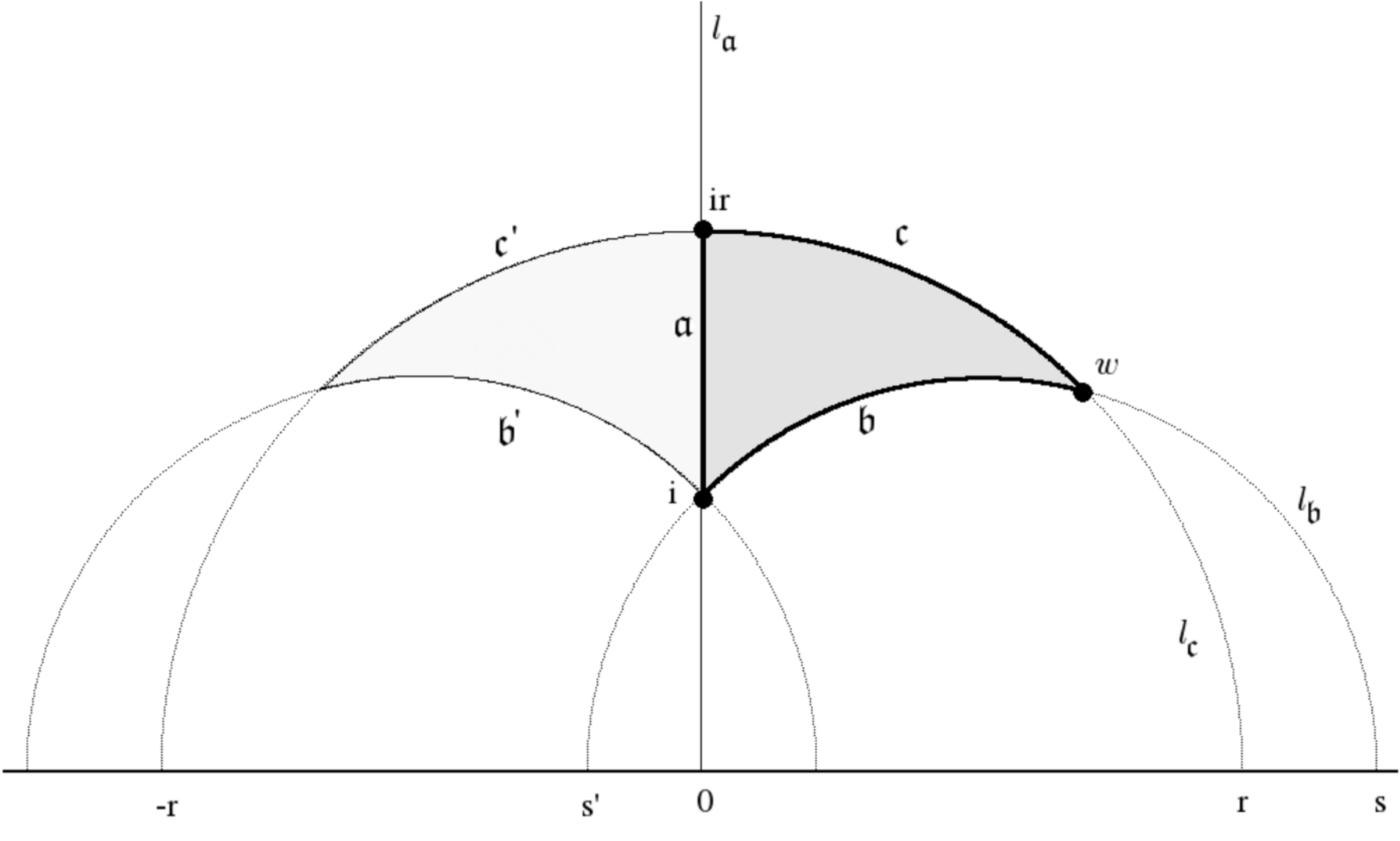
$$N(\phi_\lambda) \gg \lambda^{1/27} .$$

RECENTLY J. JANG AND J. JUNG PROVED AN UNCONDITIONAL RESULT IN THE ABOVE COMPACT X SETTING.

THEOREM 5 (JANG-JUNG 2018):

LET X BE A COMPACT ARITHMETIC SURFACE AS ABOVE AND ϕ_λ ALSO AS ABOVE THEN $N(\phi_\lambda) \to \infty$ AS $\lambda \to \infty$.

THEOREMS 4 AND 5 APPLY TO THE COMPACT ARITHMETIC TRIANGLES WITH DIRICHLET OR NEUMANN BDRY CONDITIONS. EG $(\pi/2, \pi/6, \pi/6)$



l_a
ir
c'
c
a
w
b'
b
i
l_b
l_c
-r
s'
0
r
s

THE PROOFS MAKE USE OF VARIOUS RESULTS IN THE SEMI-CLASSICS OF MAASS FORMS "QUANTUM CHAOS" AS WELL AS PERIOD FORMULAE.

(1) QUE: QUANTUM UNIQUE ERGODICITY (LINDENSTRAUSS, SOUNDARARAJAN)

ϕ_λ ON X_Γ, $a \in C(\Gamma \backslash SL_2(\mathbb{R}))$
$\cong T_1^*(X_\Gamma)$

$\|\phi_\lambda\|_2 = 1$, THEN

$$\langle Op(a)\phi_\lambda, \phi_\lambda \rangle \longrightarrow \int_{\Gamma \backslash SL_2(\mathbb{R})} a(g)\,dg \quad \text{AS } \lambda \to \infty.$$

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(2) SUBCONVEX BOUND FOR L^∞:

$$\|\phi_\lambda\|_\infty \ll \lambda^{5/24} \cdot \|\phi_\lambda\|_2$$

(IWANIEC / S) $\Gamma / \mathbb{Q}$

$$\|\phi_\lambda\|_\infty \underset{X_\Gamma}{\ll} \lambda^{1/4 - \frac{1}{27}}, \quad (\Gamma/F)$$

(BLOMER-MICHEL)

(3) WALDSPURGER'S FORMULA

$$\left| \int_{\delta} \phi(s)\,\omega(s)\,ds \right|^2 = * \, L\left(\tfrac{1}{2}, \pi \times \omega\right)$$

explicit and can be estimated

ω A CHARACTER OF THE TORUS DEFINED BY δ, π BASE CHANGE OF ϕ TO E THE CORRESPONDING QUADRATIC EXTN OF F.

- HOW TO PRODUCE NODAL DOMAINS? CAN WE LOCALIZE? NOTE THAT IN STERN'S EXAMPLE $\phi|_{\partial\square}$ HAS ONLY TWO SIGN CHANGES.

TOPOLOGICAL REDUCTION TO δ.

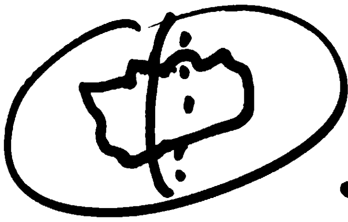
TOPOLOGICAL LEMMA (GENERAL SURFACE)

X, δ, σ AS ABOVE

$\partial_n \phi|_\delta = 0$ (IE ϕ IS σ EVEN)

LET $n(\phi)$ BE THE NUMBER OF SIGN CHANGES OF ϕ AS ONE TRAVERSES δ, THEN

$$N(\phi) \geq \frac{n(\phi)}{2} + 1 - \text{genus}(X)$$



IF $Z(\phi)$ HAS NO SELF CROSSINGS THE BOUND IS EASY TO PROVE BY INDUCTION; THE POINT WHICH WORKS IN OUR FAVOR HERE IS THAT SELF CROSSINGS ONLY INCREASE THE NUMBER OF NODAL DOMAINS.

• THIS LOCALIZES OUR PROBLEM TO PRODUCING SIGN CHANGES OF ϕ ALONG δ.

• THE NODAL DOMAINS THAT WE PRODUCE IN THIS WAY ARE ALL "σ-INERT", THAT IS THEY ARE INVARIANT UNDER σ OR WHAT IS THE SAME THING THEY CROSS δ. THEIR NUMBER IN THIS REAL ANALYTIC SETTING CAN BE BOUNDED FROM ABOVE (TOTH - ZELDITCH 2008)

$$N_{inert}(\phi_\lambda) \ll \lambda^{1/2} .$$

SO WITH THIS METHOD WE WILL NEVER PRODUCE THE EXPECTED $c\lambda$ DOMAINS MOST OF WHICH ARE SPLIT.

L^2-RESTRICTIONS TO δ (G-R-S)

IF X IS COMPACT (AND IN THE NON-COMPACT CASE SOME ARITHMETIC INPUT IS USED (AND δ BELOW IS REPLACED BY A LARG COMPACT SUBSEGMENT) THEN QUE IMPLIES THAT OUR ϕ_λ'S SATISFY

$$\int_\delta |\phi_\lambda(s)|^2 ds \gg 1 \qquad \left(\|\phi_\lambda\|_2 = 1\right)$$

(IF ϕ_λ IS NOT EVEN ABOUT δ THEN THIS IS REPLACED BY A COMBINATION OF THE L^2 RESTRICTION OF ϕ_λ TO δ PLUS THE L^2-RESTRICTION OF $\partial_n \phi$ (NORMALIZED PROPERLY).

THE IDEA TO PRODUCE SIGN CHANGES IS TO COMPARE

$$\left|\int_C \phi \, ds\right| \quad \text{WITH} \quad \int_C |\phi| \, ds$$

FOR SUITABLE SEGMENTS C OF δ AND TO SHOW THESE AREN'T EQUAL.

WE HAVE:

$$1 \ll \int_\delta |\phi(s)|^2 ds \leq \|\phi\|_\infty \int_\delta |\phi(s)| ds$$

$$= \|\phi|_\delta\|_\infty \sum_{j=1}^{n} \left|\int_{C_j} \phi_\lambda(s) \, ds\right|$$

$\longrightarrow$ segments of fixed sign.

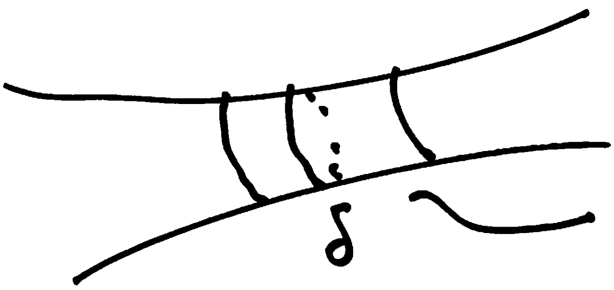
Waldpurger's formula + Lindeloff

$$\Longrightarrow \quad \int_C \phi_\lambda(s) \, ds \ll_\varepsilon \lambda^{-1/4+\varepsilon}.$$

SO THE SUBCONVEX BOUND FOR L^∞ IS PRECISELY WHAT IS NEEDED TO OBTAIN A (POWER) LOWER BOUND FOR η.

JANG-JUNG:

THEY START BY EXPLICATING IN THIS CASE AN INTEGRATION BY PARTS ARGUMENT OF CHRISTIANSON - TOTH - ZELDITCH



CONSTRUCT A CUTOFF ψ.d.op. IN A NBH OF δ

AND USE THE QUE HYPOTHESIS

LEADS TO:

IF $f(s) \geq 0$ ON δ IS FIXED, $m \geq 0$

THEN

$$\lim_{\lambda \to \infty} \left[\int_\delta |f(s)\,\phi_\lambda(s)|^2 ds - \lambda^{-m} \int_\delta \left| \partial_s^m \big(f(s)\,\phi_\lambda(s)\big) \right|^2 ds \right]$$

$$= 2\left(1 - \frac{1}{\pi} \int_{-1}^{1} \frac{\xi^{2m}\, d\xi}{\sqrt{1-\xi^2}}\right) \int_\delta f^2(s)\, ds$$

$$\xrightarrow{(*)}$$

$$\Longrightarrow \quad \varliminf_{\lambda \to \infty} \int_\delta f(s)^2 \phi_\lambda^2(s)\, ds \geq 2 \int_\delta f(s)^2\, ds$$

WHICH IS A STRONGER AND LOCALIZED IMPROVEMENT OF THE L^2-RESTRICTION BOUND ABOVE.

(18)

THERE IS NOTHING TO STOP

$$\| \phi_\lambda |_\delta \|_2 \to \infty \text{ AS } \lambda \to \infty.$$

THEY RENORMALIZE

$$\frac{f(s)\,\phi_\lambda(s)}{\| \phi_\lambda |_\delta \|_2}$$

AND STUDY THE POSSIBLE WEAK LIMITS OF THIS SEQUENCE ON δ.
THE FOURIER TRANSFORMS (ON δ)

$$\frac{\lambda\, \hat{\phi}_\lambda(\lambda \xi)}{\int_\delta |\phi_\lambda|^2_\delta}$$

ARE POSITIVE DEFINITE IF $\phi_\lambda(s) f(s)$ IS A FIXED SIGN, THEY SHOW THAT THE HAVE ENOUGH INFORMATION FROM C*A TO SHOW THIS CANT HAPPEN!