Iwasawa Theory and Bloch-Kato Conjecture for Unitary Groups

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One of the central problems in number theory is to study relation between analytic objects (e.g. special values of L-functions) and arithmetic objects. Examples include class number formula for number fields and BSD conjecture for elliptic curves. More generally Bloch-Kato (1994) formulated a Tamagawa number conjecture, giving a precise relation of the form

$$L-\text{function values} \approx \text{sizes (arithmetic) Selmer groups},$$

$$\text{Vanishing orders} = \text{Ranks of Selmer Groups}.$$

Set up

- Let K be a CM field over its maximal totally real sub-field F.
- Let p be a rational prime split completely in K.
- From now on for simplicity we assume $F = \mathbb{Q}$. So we can assume p split as $v_0\bar{v}_0$. Let K_∞ be the unique $\mathbb{Z}_p^2$ extension of K and Γ_K be the Galois group.
- Write the Iwasawa algebra $\Lambda = \mathscr{O}_L[[\Gamma_K]]$ for $\mathscr{O}_L$ the integer ring of some finite extension L of $\mathbb{Q}_p$. So $\mathrm{Spec}\Lambda$ parameterizes characters of Γ_K.

Automorphic Forms

Let $\mathrm{U}(r,s)$ be a unitary group of signature (r,s). Let $n=r+s$. Suppose π is an irreducible cuspidal automorphic representation with algebraic weight

$$\underline{k}=(c_1,\cdots,c_r;c_{r+1},\cdots,c_{r+s}).$$

The weight satisfies

$$c_1\geq\cdots\geq c_r\geq c_{r+1}\geq\cdots\geq c_{r+s},\ \ c_r\geq c_{r+1}+n.$$

Galois representations

By work of lots of people (Harris-Taylor, Shin, Morel, ...) , there is Galois representation (normalized by Geometric Frobenius)

$$\rho_\pi : G_K \to \mathrm{GL}_n(L) = \mathrm{GL}(V).$$

The Galois representation is characterized by requiring the Satake parameters equal Frobenius eigenvalues at all good primes.

- Now suppose π is unramified and ordinary at all primes above p. The notion of being ordinary is defined using the Satake parameters at p-adic places and the weight $\underline{k}$, which basically says that the eigenvalues of U_p operators are p-adic units.

Then ρ_π satisfies

$$\rho_\pi|_{G_{v_0}} \simeq \begin{pmatrix} \xi_{1,v_0}\epsilon^{\kappa_{1,v_0}} & * & * \\ 0 & \cdots & * \\ 0 & 0 & \xi_{n,v_0}\epsilon^{\kappa_{n,v_0}} \end{pmatrix},$$

and

$$\rho_\pi|_{G_{\bar{v}_0}} \simeq \begin{pmatrix} \xi_{1,\bar{v}_0}\epsilon^{\kappa_{1,\bar{v}_0}} & * & * \\ 0 & \cdots & * \\ 0 & 0 & \xi_{1,\bar{v}_0}\epsilon^{\kappa_{n,\bar{v}_0}} \end{pmatrix}.$$

Here ξ's are unramified characters. The Hodge-Tate weights κ_i's depends on $\underline{k}$ and satisfy $\kappa_1 > \kappa_2 > \cdots > \kappa_n$ for both v_0 and $\bar{v}_0$. It is well known that there is an $\mathscr{O}_L$-lattice T of V stable under G_K.

Now we make the following assumption:

(Irred) There is a Galois stable lattice T such that the resulting residual Galois representation $\bar{\rho}_\pi$ is absolutely irreducible.

Under this assumption, the Galois stable lattice T is unique up to scalar.
Let χ be a Hecke character of $K^\times\backslash\mathbb{A}_K^\times$ of Hodge-Tate weight (k_1, k_2).

Suppose $L(\rho_\pi \otimes \chi^{-1}, 0)$ corresponds to critical value of L-function (following Deligne). Then there is some i such that

$$\kappa_{i+1,v_0} \leq k_1 < \kappa_{i,v_0}, \ \kappa_{n-i+1,\bar{v}_0} \leq k_2 < \kappa_{n-i,\bar{v}_0}.$$

We assume the i is the r in the signature for the reason below. In fact this L-value can be realized via doubling method for

$$\mathrm{U}(r,s) \times \mathrm{U}(s,r) \hookrightarrow \mathrm{U}(n,n),$$

where for the Siegel Eisenstein series $E_{\mathrm{sieg},\chi}$ on $\mathrm{U}(n,n)$,

$$E_{\mathrm{sieg},\chi}|_{\mathrm{U}(r,s)\times\mathrm{U}(s,r)} \sim \sum_\pi L(\rho_\pi \otimes \chi^{-1}, 0)\pi \boxtimes \tilde{\pi}.$$

In fact a recent work of Eischen-Harris-Li-Skinner constructed the p-adic L-function $\mathscr{L}_{\pi,\chi,K}$ which is an element in $\mathscr{O}_L[[\Gamma_K]] \otimes L$, interpolating algebraic part of values $L(0, \rho_\pi \otimes \chi^{-1}\chi_\phi^{-1})$ (meaning appropriate period factors are divided out), where χ_ϕ running over characters of Γ_K making the corresponding L-values critical.

Remark

The EHLS work proves the interpolation formula to the right of the center of the critical line. We will see later on that our result completes this to all critical values.

Arithmetic Side

Now we turn to the arithmetic side. Fix a finite set of primes Σ including all bad primes and primes above p. We define the Selmer group of $\rho_\pi \otimes \chi^{-1}$ over K_n between K and K_∞:

$$\mathrm{Sel}(K_n, V/T(1) \otimes \chi^{-1}) := \mathrm{Ker}\{H^1(K_n^\Sigma, V/T(1) \otimes \chi^{-1})$$
$$\to \prod_{v \in \Sigma} \frac{H^1(K_{n,v}, V/T(1) \otimes \chi^{-1})}{H^1_f(K_{n,v}, V/T(1) \otimes \chi^{-1})}\}.$$

where the H^1_f are defined as follows.

- For primes $v \nmid p$, we define
$$H^1_f(K_{n,v}, V(1) \otimes \chi^{-1}) := \ker\{H^1(K_{n,v}, V(1) \otimes \chi^{-1}) \\ \to H^1(I_{n,v}, V(1) \otimes \chi^{-1})\},$$
and $H^1_f(K_{n,v}, V/T(1) \otimes \chi^{-1})$ is defined to be the image of $H^1_f(K_{n,v}, V(1) \otimes \chi^{-1})$.
- For primes above p, recall the local Galois representation T is upper-triangular. There is a co-torsion free rank r submodule $T^+_{v_0} \subseteq T$ corresponding to the upper r rows at v_0 which is stable under G_{v_0}. Similarly there is a rank s co-torsion free submodule $T^+_{\bar{v}_0} \subseteq T$ corresponding to the upper s rows at $\bar{v}_0$. We define $H^1_f(K_{n,v_0}, V/T(1) \otimes \chi^{-1})$ as the image of $H^1(K_{n,v_0}, V^+/T^+(1) \otimes \chi^{-1})$, and similarly for $\bar{v}_0$.

The above definition is due to Greenberg. We define

$$\mathrm{Sel}(K_\infty, V/T(1)\otimes\chi) = \varinjlim_{K_n} \mathrm{Sel}(K_n, V/T(1)\otimes\chi^{-1}),$$

and $X_{\pi,\chi}$ being its Pontryagin dual. We define characteristic ideal as follows.

Definition

Let A is a Noetherian normal domain and M a finitely generated A-module. We define the characteristic ideal $\mathrm{char}_A(M)$ to consist of element $x \in A$ such that for all height one primes P of A, the length of M_P as an A_P module is less than or equal to the valuation of x at P. If M is not torsion then the characteristic ideal is 0. The notion of characteristic ideal measures the size of the module M.

Conjecture

(Greenberg's Iwasawa main conjecture)(vague) The $X_{\pi,\chi}$ *is a torsion module over* Λ*, and its characteristic ideal is generated to the p-adic L-function constructed by Eischen-Harris-Li-Skinner.*

Remark

In fact there is still ambiguity since EHLS did not make a nice choice for period so as to make the p-adic L-function integral.

Now we give the main theorem. We cannot quite prove (or even formulate) the precise main conjecture, but can prove the weaker result below.

Theorem

Assume (Irred), π being ordinary at p and a mild assumption on the weight. If $L(\frac{1}{2}, K, \pi \otimes \chi^{-1}) = 0$, then the co-rank of $\mathrm{Sel}(K, V/T(1) \otimes \chi^{-1})$ is at least one.

Remark

If $F = \mathbb{Q}$ and the motive R (i.e. $\rho_\pi(1) \otimes \chi^{-1}$) satisfies $R = R^{\vee,c}(1)$, then it is proved in Skinner-Urban's ICM 2006 report. If in addition assume the global sign is -1, it is proved by Bellaiche-Chenevier assuming Authur's conjecture. In these work the proofs are *not* Iwasawa theoretic (i.e. not involving the variable of cyclotomic twists).

Remark

For example when $F = \mathbb{Q}$, twisting R by finite order cyclotomic characters may break the assumption of Skinner-Urban (ICM 2006) and Bellaiche-Chenevier, but OK for our assumption.

Outline for Eisenstein Congruence Argument

We first sketch the proof.

- We first construct p-adic families of ordinary Klingen Eisenstein series on $\mathrm{U}(r+1, s+1)$ whose Galois representation is ρ_π plus two characters. We prove the constant terms are divisible by the p-adic L-function we study (interpolating Langlands-Shahidi). This is essentially done in my thesis. (In fact we need also deform π in Hida families).
- Next, ideally we hope to prove that certain Fourier-Jacobi coefficient for this Eisenstein family is co-prime to the p-adic L-function we study. In other words the Eisenstein family constructed is "primitive". Unfortunately we can only prove weaker versions of it which is good for the theorem above. (This is the technical part of the whole argument).

- The above step plus some geometric argument of Hida theory tells us the Klingen Eisenstein family is congruent to some cuspidal Hida family modulo the p-adic L-function. Use this congruence, passing to the Galois side, a Galois theoretic argument called lattice construction (generalized "Ribet lemma" by Wiles, Urban and Bellaiche-Chenevier, etc) provides enough elements in the Selmer group.

Remark

The other bound for Selmer groups are usually studied using Euler systems, which we do not discuss here.

Remark

Mazur-Wiles (1984) and Wiles (1990) used the congruences between Eisenstein series and cusp forms on GL_2 to deduce the Iwasawa main conjecture for Hecke characters (i.e. GL_1) over totally real fields. Urban used the Eisenstein congruences on Gsp_4 to study Iwasawa main conjecture for adjoint L-functions for modular forms.

Remark

The unitary groups are studied in cases when (r, s) is $(1, 0)$ (by M-L Hsieh) or $(1, 1)$ (by Skinner-Urban when $F = \mathbb{Q}$ and myself in general) or $(2, 0)$ by myself. In these cases stronger results (namely the one divisibility of main conjecture) are proven. These resulted in recent important progresses for p-part of BSD formulas (or Tamagawa number conjectures in general), and converse of Gross-Zagier and Kolyvagin theorem by Skinner and Zhang.

Remark

In the work of Skinner-Urban (ICM 2006) and Bellaiche-Chenevier, they need to assume $F = \mathbb{Q}$ and $R = R^{\vee,c}(1)$ since otherwise in the lattice construction, there is a positive rank Selmer group of Hecke character interacting with the extension classes in the Selmer group we are studying, which is hard to separate out. Our argument uses Iwasawa theory of ordinary Eisenstein families, and can thus further study the "multiplicities" along cyclotomic twist direction. Skinner-Urban worked with Eisenstein series of critical slope, which does not form a nice family along the cyclotomic direction.

Remark

We see

$$\begin{pmatrix} \chi_1 & *_1 & *_2 \\ & \rho_\pi & \\ & & \chi_2 \end{pmatrix}$$

where the $*_1$ is the Galois extension class we look for, and $*_2$ is the Galois extension class for Hecke characters. There is some interaction between $*_1$ and $*_2$ in the lattice construction. In Skinner-Urban's situation, the $*_2$ has rank 0 by Kummer theory and that the unit group $\mathcal{O}_K^\times$ has rank 0. In general the $*_2$ may have positive rank.

We explain a little more comparison between our strategy and Skinner-Urban's. By Langlands-Shahidi computation, the constant term of the Klingen Eisenstein series satisfies

$$E_{\mathrm{const}}(g,z) = L_1 f_z + L_2 M(f_{-z}, z)$$

where f_z is the section used to define the Eisenstein series, the $M(f,z)$ is the intertwining operator, and L_1 and L_2 are ratios of some special values of L-functions. In Skinner-Urban's ICM 2006 case, at the point of study the L_2 involves the central critical value, at which the Klingen Eisenstein series is a classical form. In our case, at the point we study, the L_1 involves the central critical value.

- In those previous work on low rank cases ($\mathrm{U}(1,1)$, $\mathrm{U}(2,0)$ and $\mathrm{U}(1,0)$) we compute the Fourier-Jacobi expansion of the Klingen Eisenstein family to obtain the primitivity results – namely prove they are coprime to the p-adic L-function we study. In thoses cases an appropriate Fourier-Jacobi coefficient can be expressed as a product of (CM type, Rankin-Selberg or triple product) p-adic L-functions. However for general $\mathrm{U}(r,s)$ the automorphic meaning for FJ coefficient is unclear.
- Our proof here is different, do not explicitly compute Fourier-Jacobi coefficients.

Proof for Non-vanishing

- Instead of proving the Klingen Eisenstein family is coprime to the p-adic L-function, we only prove that it is coprime to any height one prime passing through the arithmetic point ϕ corresponding to the vanishing central L-value we study. This is enough for proving our weaker result. Thus we are reduced to proving that the specialization of the Eisenstein family to ϕ is non-vanishing.
- This specialization is only a p-adic limit since it has "negative weight". Thus we can not simply invoke the classical automorphic theory. This is the main difficulty.

The Idea

- The key idea is to use the *p-adic functional equation* for the Eisenstein family and p-adic L-functions.
- We consider the Maass-Shimura differential operator $\delta_{r+1,s+1}$ acting on the nearly holomorphic or p-adic automorphic forms on $\mathrm{U}(r+1,s+1)$. Recall $E^{\mathrm{ord}}_{\mathrm{Kling},\phi}$ does not have classical weight (ϕ is the arithmetic point where the constant term corresponds to the central L-value). However the $\delta_{r+1,s+1}E^{\mathrm{ord}}_{\mathrm{Kling},\phi}$ *does* have classical weight.
- We use the functional equation to show $\delta_{r+1,s+1}E^{\mathrm{ord}}_{\mathrm{Kling},\phi}$ is nonzero, which implies $E^{\mathrm{ord}}_{\mathrm{Kling},\phi}$ is nonzero.

p-adic Functional equation

- The p-adic functional equation identifies the $\delta_{r+1,s+1}E^{\mathrm{ord}}_{\mathrm{Kling},\phi}$ with the p-adic avatar of a classical Klingen Eisenstein series $E^{\mathrm{crit}}_{\mathrm{Kling},\phi+(1,\cdots,1;-1,\cdots,-1)}$.
- In fact the E_{Kling} is realized via Shimura's pullback formula

$$\mathrm{U}(r+1,s+1)\times\mathrm{U}(s,r)\hookrightarrow\mathrm{U}(r+s+1,r+s+1)$$

of the Siegel Eisenstein series E_{Sieg} on $\mathrm{U}(r+s+1,r+s+1)$. We first establish a p-adic functional equation for E_{Sieg}.

We give a toy example in the GL_2 case used by Katz for CM L-functions. For natural numbers a and b, consider

$$\sum_{p\nmid n}\sum_{d|n} d^a\left(\frac{n}{d}\right)^b q^n = \sum_{p\nmid n}\sum_{d|n} d^{a-b}n^b q^n = \delta^b\left(\sum_{p\nmid n}\sum_{d|n} d^{a-b}q^n\right) := \delta^b E_{a,b}.$$

Here δ is the Maass-Shimura differential operator, and $E_{a,b}$ is a holomorphic Eisenstein series. When varying a and b p-adically, there is clearly a symmetry for a and b. This is the p-adic functional equation for GL_2 Eisenstein series.

Remark

The p-adic variation (as a, b above) can be viewed as analogue of the parameter z for classical Eisenstein series.

For $\mathrm{U}(r,s)$ we first need a higher rank analogue of Katz. We obtain such symmetry for formal q-expansion of the Siegel Eisenstein series on $\mathrm{U}(r+s+1, r+s+1)$ from

- A result of Kudla-Sweet on local functional equation for non-degenerate Whittaker coefficients of degenerate principal series. In the GL_2 case this is elementary

$$(1+q^{-s}+\cdots+q^{-ns})q^{ns} = 1+q^{s}+\cdots+q^{ns}.$$

- A computation of p-adic differential operators used extensively in the construction of the Eisenstein family by Eischen-Harris-Li-Skinner.

Non-vanishing

In the next step we prove the $E^{\mathrm{crit}}_{\mathrm{Kling},\phi+(1,\cdots,1;-1,\cdots,-1)}$ is nonzero by explicit computations using classical automorphic theory (local computations). This is the *primitivity* result we need for the Eisenstein family, and is the technical part of the whole argument (involving lot of computations).
Once with this we conclude that $E^{\mathrm{ord}}_{\mathrm{Kling},\phi}$ is also nonzero since $E^{\mathrm{crit}}_{\mathrm{Kling},\phi+(1,\cdots,1;-1,\cdots,-1)}$ is $\delta_{r+1,s+1}E^{\mathrm{ord}}_{\mathrm{Kling},\phi}$.

Remark

Along the way we also obtain the p-adic functional equation for p-adic L-functions of Eischen-Harris-Li-Skinner, proving the interpolation formula at all critical values (instead of only to one side of the central strip).

Remark

It seems the method can be generalized to finite slope cases, using "trianguline Iwasawa theory" (Pottharst), and some construction of finite slope families generalizing a recent work of Andreatta-Iovita from GL_2 to unitary group case. This is joint work in progress with F. Castella and Z. Liu.

Thanks

Thank You !