Branching laws for representations of real reductive groups

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Branching problem

Let G be a group and G' a subgroup.

Let V be an irreducible representation of G.

Decompose the restriction $V|_{G'}$ into irreducible objects.

Suppose that G and G' are real linear reductive Lie groups.

i.e. $G \subset GL(N, \mathbb{R})$: closed, finitely many conn. components,

and ${}^tG = G$, same for G'.

e.g. $G = SL(n, \mathbb{R})$, $Sp(n, \mathbb{C})$, $O(p, q)$, $E_{6(2)}$, $\cdots$

$G \supset G'$ real linear reductive Lie groups

$\mathcal{H}$ an irreducible unitary representation of G

$$\Rightarrow \mathcal{H}|_{G'} = \underbrace{\sum_{\tau \in \widehat{G'}}^{\oplus} m_d(\tau)\, \mathcal{H}_\tau}_{\text{discrete part}} \oplus \underbrace{\int_{\tau \in \widehat{G'}}^{\oplus} m_c(\tau) \mathcal{H}_\tau \, \mu(\tau)}_{\text{continuous part}}$$

If G' is compact, we have $\mathcal{H}|_{G'} = \sum_{\tau \in \widehat{G'}}^{\oplus} m_d(\tau)\, \mathcal{H}_\tau$.

Not much is known about explicit branching laws for non-compact G'.

$(\mathfrak{g}, K)$-modules

$\mathfrak{g} := \mathrm{Lie}\ G$,

K a maximal compact subgroup of G, $\quad K = G \cap O(N)$

$(\mathfrak{g}, K)$-module $=$ $\mathfrak{g}$-action $+$ locally finite K-action

$+$ compatibility condition

$\{$ irred. unitary rep. of G $\} \longleftrightarrow \{$ irred. unitarizable $(\mathfrak{g}, K)$-mod. $\}$

$$\mathcal{H} \quad \longrightarrow \quad \mathcal{H}_K$$

$$\mathcal{H}_K := \{v \in \mathcal{H} : \dim\langle Kv\rangle < \infty\}$$

$$\overline{V} \quad \longleftarrow \quad V$$

Hilbert completion

Discrete decomposability of $(\mathfrak{g}, K)$-modules (Kobayashi '90s)

$G \rightsquigarrow (\mathfrak{g}, K), \qquad G' \rightsquigarrow (\mathfrak{g}', K')$

Consider branching problem for $(\mathfrak{g}, K)$-modules.

Definition Let V be a unitarizable $(\mathfrak{g}, K)$-module.

$V|_{\mathfrak{g}'}$ is discretely decomposable

$\iff V = V_1 \oplus V_2 \oplus \cdots$ as a $(\mathfrak{g}', K')$-module.

(V_i : an irreducible $(\mathfrak{g}', K')$-module)

If $V|_{\mathfrak{g}'} = \bigoplus_i V_i$, then $\overline{V}|_{G'} = \hat{\bigoplus}_i \overline{V_i}$.

If we assume the discrete decomposability, the branching problems for unitary representations of groups are equivalent to the branching problems for corresponding $(\mathfrak{g}, K)$-modules.

Zuckerman's derived functor module

$\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ a parabolic subalgebra such that ${}^t\mathfrak{q} = \mathfrak{q}$

λ a character of $\mathfrak{q}$

$\rightsquigarrow \quad A_{\mathfrak{q}}(\lambda)$: $(\mathfrak{g}, K)$-module (cohomological induction)

Theorem (Zuckerman, Vogan, Wallach)

Under certain positivity condition on λ, the $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ is irreducible and unitarizable.

Remark The discrete seires representations are isomorphic to $A_{\mathfrak{q}}(\lambda)$ for some $\mathfrak{q}$ and λ.

Suppose that (G, G') is a symmetric pair.

i.e. $\exists\sigma$ an involution of G such that $G' = \{g \in G : \sigma(g) = g\}$.

e.g. $(G, G') = (H \times H, \Delta H)$, $(U(p,q), U(p,q-1) \times U(1))$, $(SL(p+q, \mathbb{R}), SO(p,q))$, $\cdots$.

Theorem (Kobayashi)

$A_{\mathfrak{q}}(\lambda)|_{\mathfrak{g}'}$ is discretely decomposable

$$\iff \quad \mathrm{pr}_{\mathfrak{g}\downarrow\mathfrak{g}'}\bigl(\underline{\mathrm{Ass}_{\mathfrak{g}}(A_{\mathfrak{q}}(\lambda))}\bigr) \subset \mathcal{N}(\mathfrak{g}'_{\mathbb{C}}).$$

associated variety

Classification (Kobayashi–O.)

Branching law

Theorem (O. 2013)
Let (G, G') be a symmetric pair of real reductive groups and assume that $A_{\mathfrak{q}}(\lambda)|_{\mathfrak{g}'}$ is discretely decomposable.
Then

$$A_{\mathfrak{q}}(\lambda)|_{\mathfrak{g}'} \;=\; \bigoplus_{i} m_{1,i}\, A_{\mathfrak{q}'_1}(\lambda_i) \oplus \bigoplus_{j} m_{2,j}\, A_{\mathfrak{q}'_2}(\lambda_j) \oplus \cdots$$

Proof uses $\mathcal{D}$-module realization of $A_{\mathfrak{q}}(\lambda)$ and case-by-case analysis.