High dimensional expanders

Ori Parzanchevski

IAS, Oct 1, 2013

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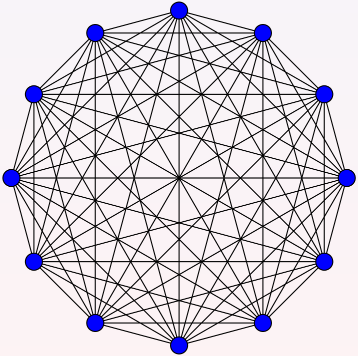

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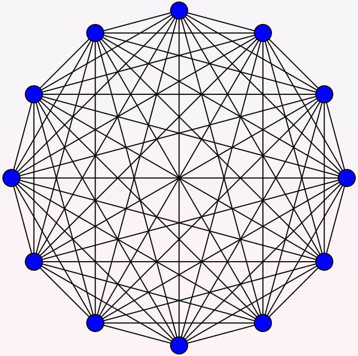

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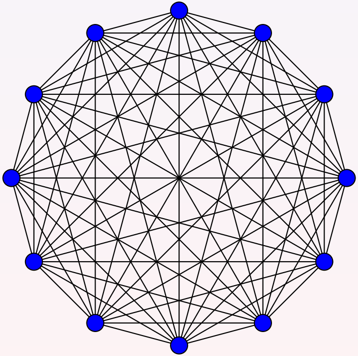

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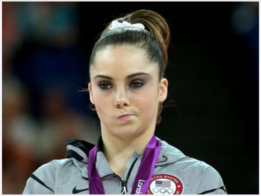

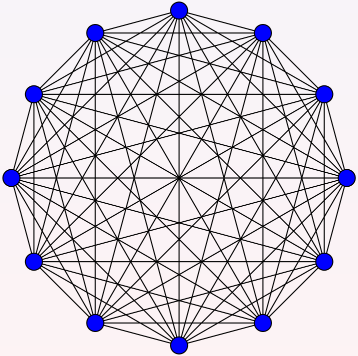

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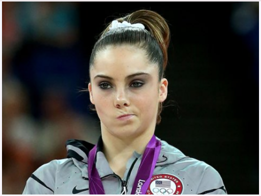

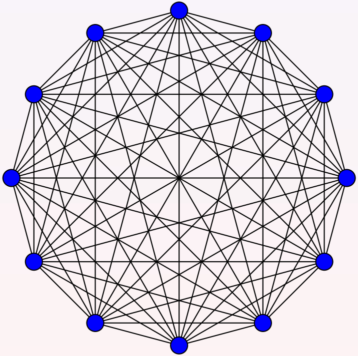

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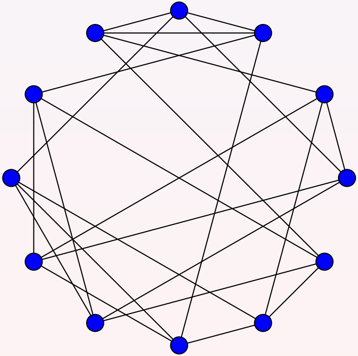
Better:



Similar properties, but now only 2 edges per vertex!

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Expansion in graphs

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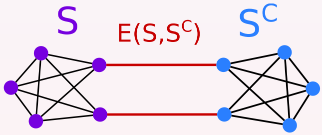

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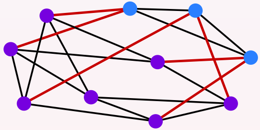

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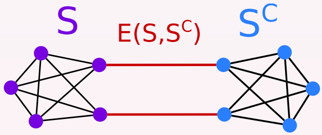

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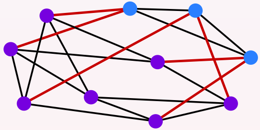

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- Hard to analyze $h(G)$.

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Corollary: for k-regular graphs, $h > c > 0$ iff $\lambda > c' > 0$ (expanders).

Geometry (Prehistory)

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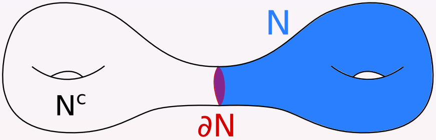

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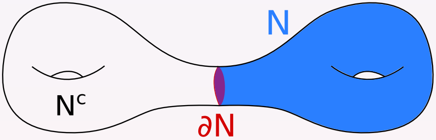

Theorem (Maz'ya '62, Cheeger '70, Buser '82)

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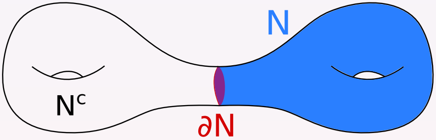

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So: for manifolds with globally bounded curvature, $h > c > 0$ iff $\lambda > c' > 0$.

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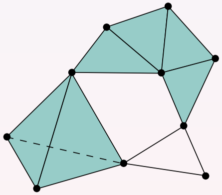

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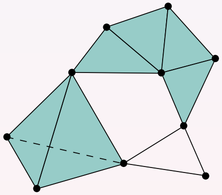

- One assumption: $\tau \subset \sigma \in X \Rightarrow \tau \in X$

Expansion in complexes

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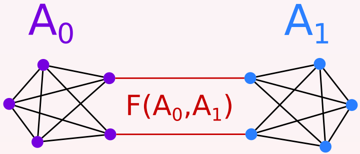

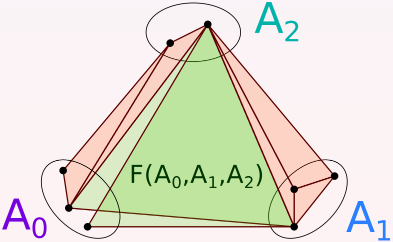

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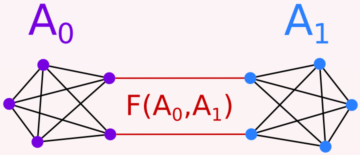

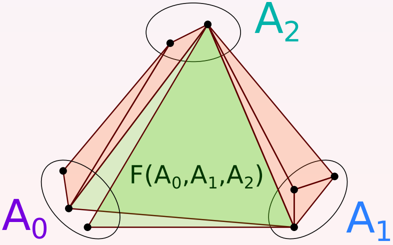

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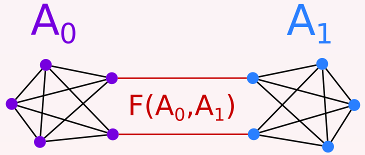

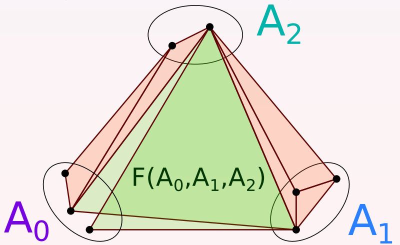

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6. Random walks

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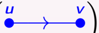
$f \in \Omega^0 = \mathbb{R}^V$: $(\delta f)\Big($ 

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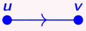
$f \in \Omega^0 = \mathbb{R}^V$: $(\delta f)\Big($  $\Big) = f(u) - f(v)$

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- $H^j = \dfrac{\ker \delta : \Omega^j \to \Omega^{j+1}}{\operatorname{im} \delta : \Omega^{j-1} \to \Omega^j}$ is the j^{th} cohomology of X (over $\mathbb{R}$) (Poincaré '95).

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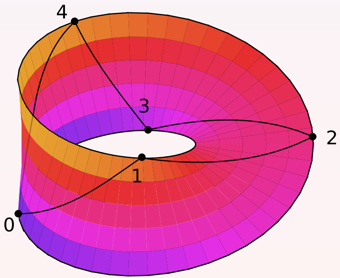

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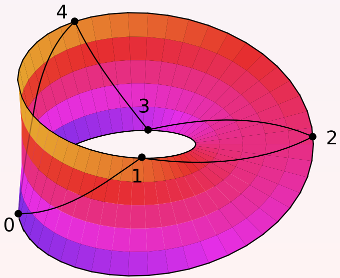

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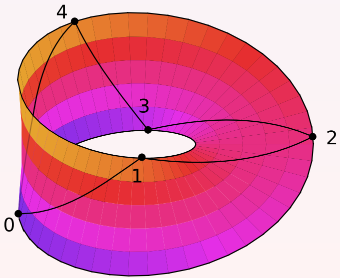

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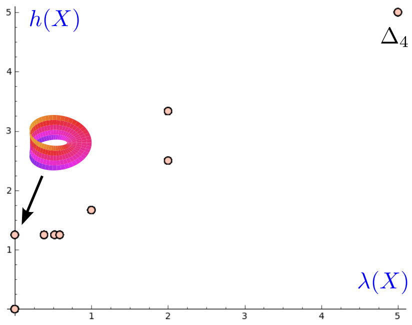

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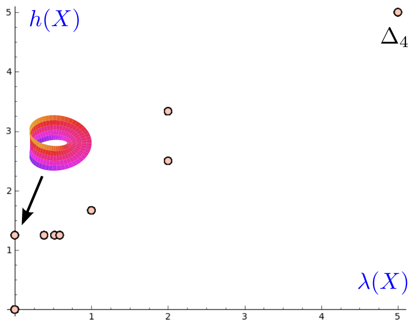

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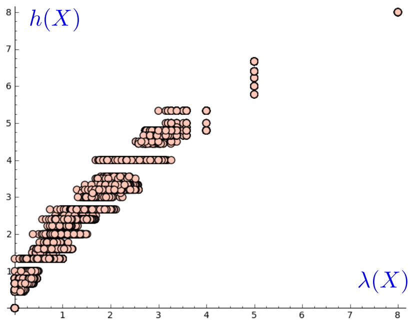

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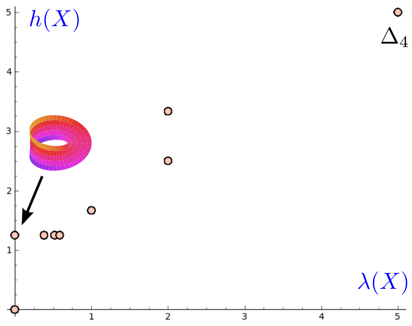

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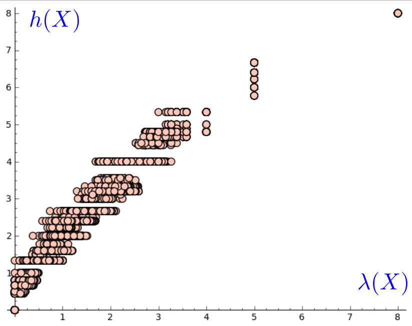

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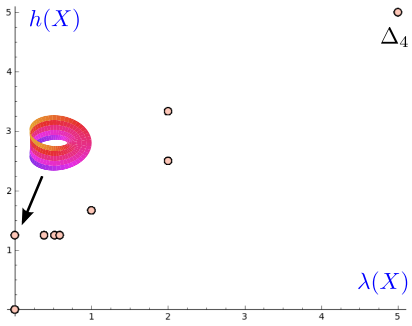
If X is a d-complex with a complete skeleton, then $\lambda(X) \leq h(X)$.



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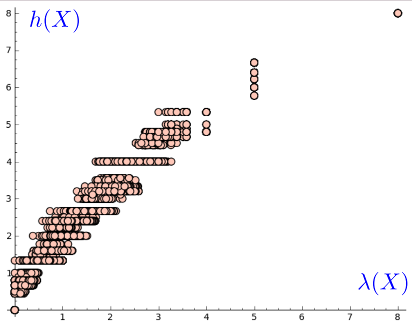

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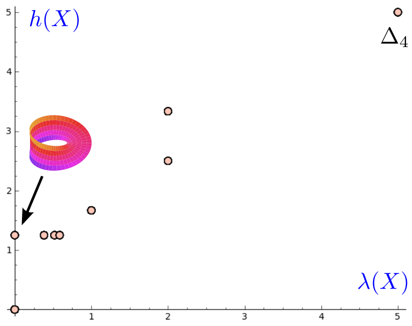

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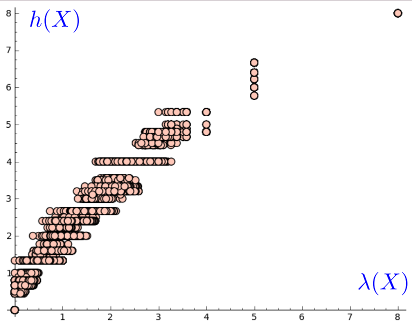

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(Generalizes the "Expander Mixing Lemma" of Friedman−Pippenger '87, Alon−Chung '88, Beigel−Margulis−Spielman '93)

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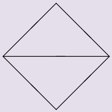

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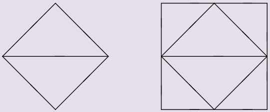

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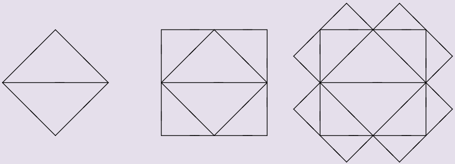

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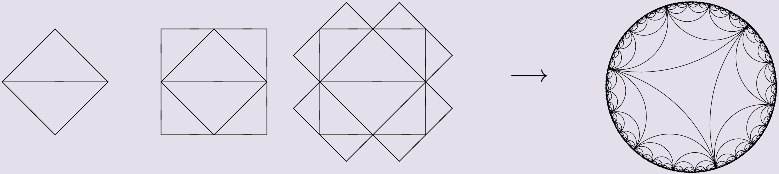

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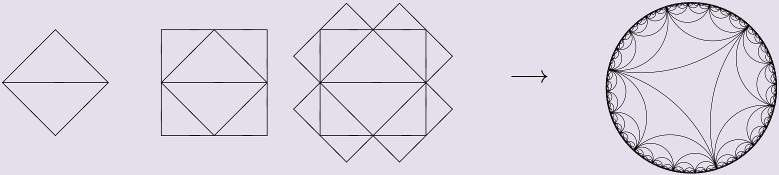

$\lim \lambda(X_n) = 3 - 2\sqrt{2}$, *whereas* $\operatorname{Spec} \Delta(X) \big|_{Z_1} = \{0\} \cup \left[3 - 2\sqrt{2}, 3 + 2\sqrt{2}\right]$.

Thank You!