Diffusion and super-diffusion of energy in one dimensional systems of oscillators

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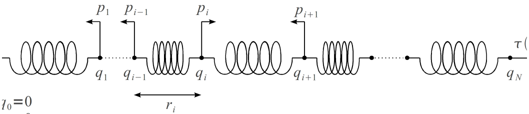
Chain of oscillators with tension



Hamiltonian dynamics (FPU type):

$$\dot{r}_j(t) = p_j(t) - p_{j-1}(t), \qquad j = 1, \dots, N,$$
$$dp_j(t) = \big(V'(r_{j+1}(t)) - V'(r_j(t))\big)\, dt, \qquad j = 1, \dots, N-1,$$
$$dp_N(t) = \big(\tau_1(t/N) - V'(r_N(t))\big)\, dt,$$

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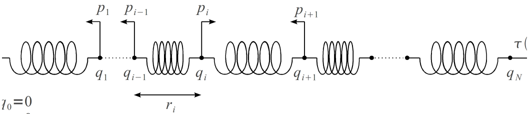

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\end{aligned}
$$

we add a noise that conserve energy and momentum:

momentum exchange For each couple of nearest neighbor particle, we randomly exchange momentum, $(p_i, p_{i+1}) \to (p_{i+1}, p_i)$, with intensity 1.

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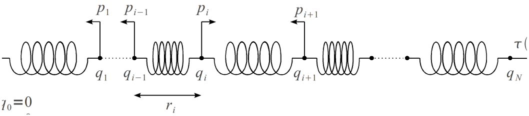

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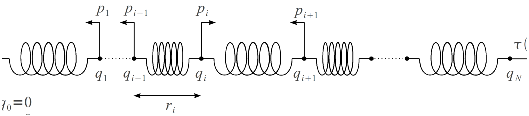
Chain of oscillators: infinite model



Hamiltonian dynamics (FPU type): $j \in \mathbb{Z}$

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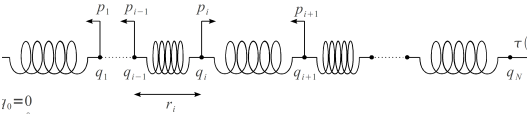

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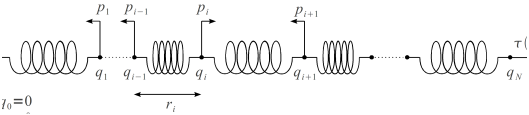

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Ergodicity (of the infinite system)

The stocastic perturbation of the dynamics is sufficient to give ergodicity:

Theorem

(Fritz, Funaki, Lebowitz, PTRF 1994)
Assume that ν *is translation invariant and stationary, with finite entropy density, and furthemore that*

$$\nu(dp|r)$$

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- $\nu(dp|r)$ maxwellian (Gallavotti-Verboven 1975)
- $\nu(dp|r)$ convex combination of maxwellians (Olla, Varadhan, Yau, 1993).

Hyperbolic Adiabatic Dynamics

3 conserved quantities:

stretch $\mathcal{R}_N(t)[G] = \frac{1}{N}\sum_i G(i/N) r_i(Nt)$

momentum $\pi_N(t)[G] = \frac{1}{N}\sum_i G(i/N) p_i(Nt)$

energy $\mathfrak{e}_N(t)[G] = \frac{1}{N}\sum_i G(i/N) \mathcal{E}_i(Nt)$

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$$(\mathcal{R}_N(t), \pi_N(t), \mathfrak{e}_N(t)) \longrightarrow (r(x,t)dx, \pi(x,t)dx, \mathfrak{e}(x,t)dx)$$

$$\boxed{\begin{array}{lll} \partial_t r = \partial_x \pi & \partial_t \pi = \partial_x \tau & \partial_t \mathfrak{e} = \partial_x(\tau\pi) \\ \pi(0,t) = 0, & \tau(r(1,t), U(1,t)) = \tau_1(t) & \end{array}}$$

$U = \mathfrak{e} - \pi^2/2$: internal energy. For smooth solutions this is proven in:

- N. Even, S.O., arXiv.org/abs/1009.2175 (2010)
- S.O., SRS Varadhan, HT Yau, CMP (1993)

entropy evolution

$$\partial_t r = \partial_x \pi \qquad \partial_t \pi = \partial_x \tau \qquad \partial_t \mathfrak{e} = \partial_x(\tau\pi)$$
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$$\frac{d}{dt} S(r(x,t), U(x,t)) = 0 \qquad \text{for smooth solutions}$$

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When shocks will appear, we will have dissipation and

$$\frac{d}{dt}\int_0^1 S(r(x,t), U(x,t))dx > 0$$

and (eventually) convergence to equilibrium.

Energy Superdiffusion

Consider the Harmonic chain with noise conserving energy and momentum.
$V(r) = \frac{r^2}{2}$, $\tau(r, U) = r$, $S = 1 + log(\pi\beta^{-1})$
In the hyperbolic space-time scale limit we have

$$\begin{array}{|ccc|} \partial_t r = \partial_x \pi & \partial_t \pi = \partial_x r & \partial_t \mathfrak{e} = \partial_x (r\pi) \\ \pi(0,t) = 0, & r(1,t) = \tau_1(t) & \end{array}$$

linear wave equation: no shocks! no dissipation!
Profiles of temperature and entropy do not change in time.

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linear wave equation: no shocks! no dissipation!
Profiles of temperature and entropy do not change in time.
Since microscopically the noise made the dynamic 'ergodic', the convergence to equilibrium should happen in another space-time scale.

Diverging thermal conductivity

$$d\mathcal{E}_i = \left(\mathrm{j}_{i-1,i} - \mathrm{j}_{i,i+1}\right) dt \qquad \mathrm{j}_{i,i+1} = -p_i V'(r_{i+1}) = -p_i r_{i+1}$$

Green-Kubo formula:

$$\boxed{\mathcal{D} = \frac{2\beta^2}{3} \int_0^\infty \sum_i \left\langle \mathrm{j}_{i,i+1}(s)\mathrm{j}_{0,1}(0)\right\rangle_\beta \, ds}$$

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More generally in dimension d

$$\sum_i \left\langle \mathrm{j}_{i,i+1}(s) \mathrm{j}_{0,1}(0) \right\rangle_\beta \sim t^{-d/2}$$

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This imply, in $d = 1$, a superdiffusion of the energy: diffusive space-time scale is not the right one.

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in **pinned** case

$$\sum_i \left\langle \mathrm{j}_{i,i+1}(s)\mathrm{j}_{0,1}(0)\right\rangle_\beta \sim t^{-d/2+1}$$

diffusive cases

In $d \geq 3$ or pinned systems, diffusivity is finite and we can prove the *linearized heat equation* (*Basile, O., 2013*):

$$\boxed{\frac{\varepsilon^{-1}}{\chi(\beta)}\left[< \mathcal{E}_{[\varepsilon^{-1}x]}(\varepsilon^{-2}t)\mathcal{E}_{[0]}(0) >_\beta \; - \; \bar{e}^2\right] \underset{N\to\infty}{\longrightarrow} \frac{e^{-\frac{x^2}{2t\mathcal{D}}}}{\sqrt{2\pi\mathcal{D}t}}}$$

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In $d = 2$ unpinned (*some details in the proof still missing*):

$$\boxed{\frac{\varepsilon^{-1}}{\chi(\beta)}\left[<\mathcal{E}_{[\varepsilon^{-1}x]}\left(\frac{\varepsilon^{-2}t}{(\log\varepsilon^{-1})^2}\right)\mathcal{E}_{[0]}(0)>_\beta \;-\; \bar{e}^2\right] \underset{N\to\infty}{\longrightarrow} \frac{e^{-\frac{x^2}{2t\mathcal{D}_0}}}{\sqrt{2\pi\mathcal{D}_0 t}}}$$

Superdiffusion in $d = 1$

We expect a Levy type superdiffusion:

$$\boxed{\frac{\varepsilon^{-1}}{\chi(\beta)}\left[< \mathcal{E}_{[\varepsilon^{-1}x]}(\varepsilon^{-3/2}t)\mathcal{E}_{[0]}(0) >_\beta \; - \; \bar{e}^2\right] \underset{N\to\infty}{\longrightarrow} f(x,t)}$$

with

$$\widehat{f}(\eta, t) \;\sim\; e^{-\lambda|\eta|^{3/2}t}$$

i.e.

$$\partial_t f \;=\; -(-\Delta)^{3/4} f$$

Result: (*Komorowski, O.* 2013)
Starting with initial distribution with finite energy, the correponding Wigner distribution $W(\varepsilon^{-1}x, k, \varepsilon^{-3/2}t)$ converges to the solution of the fractional heat equation $f(x,t)$ (for a.e. k).

Conjectured result for FPU dynamics

Recent *fluctuation hydrodynamics and mode coupling* calculation by Herbert Spohn (arXiv:1305.6412, (2013)) for the deterministic FPU dynamics:

Heat mode superdiffusion:

- If V asymmetric (α-FPU), or symmetric but $\tau \neq 0$:

$$\widehat{f}(\eta, t) \ \sim \ e^{-\lambda|\eta|^{5/3} t}$$

- If V symmetric (β-FPU), and $\tau = 0$:

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$\Longrightarrow$ Heat mode in our model behave as in the symmetric FPU non-linear case.

Wave function and Wigner distribution

$r_y = q_y - q_{y-1}, \; y \in \mathbb{Z}$:

$$\begin{aligned} \dot{q}_x(t) &= p_x(t) \\ dp_x(t) &= \Delta q_x \qquad dt \; + \gamma^{1/2} \sum_{\ell=-1,0,1} Y_{x+\ell} p_x(t) \circ dw_{x+k}(t), \end{aligned}$$

$\{w_x(t)\}_{x\in\mathbb{Z}}$ iid Wiener processes,

$$Y_x = (p_x - p_{x+1})\partial_{p_{x-1}} + (p_{x+1} - p_{x-1})\partial_{p_x} + (p_{x-1} - p_x)\partial_{p_{x+1}}$$

Wave function and Wigner distribution

$r_y = q_y - q_{y-1}$, $y \in \mathbb{Z}$:

$$\begin{aligned} \dot{q}_x(t) &= p_x(t) \\ dp_x(t) &= -(\alpha * q(t))_x dt \ + \gamma^{1/2} \sum_{\ell=-1,0,1} Y_{x+\ell} p_x(t) \circ dw_{x+k}(t), \end{aligned}$$

α_x symmetric, compact support or $|\alpha_x| \le Ce^{-|x|/C}$, $\hat{\alpha}(k) > 0$ and

- $\hat{\alpha}(0) = 0$ unpinned chain
- $\hat{\alpha}''(0) > 0$ acustic chain

Dispersion function:

$$\omega(k) = \hat{\alpha}(k)^{1/2} \qquad\qquad (= 2|\sin(\pi k)|), \qquad k \in \mathbb{T}.$$

This implies

$$\int_{|k|\le\delta} \frac{\omega'(k)^2}{k^2} dk = +\infty$$

Wave function

$$\hat{f}(k) = \sum_{y \in \mathbb{Z}} f(y) e^{-i2\pi yk}$$

Complex wave function in Fourier space:

$$\boxed{\hat{\psi}(t,k) := \omega(k)\hat{q}(t,k) + i\hat{p}(t,k)}$$

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$$\begin{aligned} d\hat{\psi}(t,k) :=& -i\omega(k)\hat{\psi}(t,k)dt - \gamma\frac{\hat{\beta}(k)}{4}[\hat{\psi}(t,k) - \hat{\psi}(t,-k)^*]dt \\ &+ \gamma^{1/2} i \int_{\mathbb{T}} r(k,k')[\hat{\psi}(k-k') - \hat{\psi}(-k+k')^*]dW(t,k') \end{aligned}$$

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$$< dW^*(t,k)dW(s,k') >= \delta(k-k')\delta(t-s)dt\, dk, \quad < dW(t,k)dW(s,k') >= 0$$

$$r(k,k') = 2r_1^2(k)r_1(2(k-k')) + 2r_1(2k)r_1^2(k-k'), \qquad r_1(k) = \sin(\pi k)$$

$$\hat{\beta}(k) = 8r_1^2(k) + 4r_1^2(2k) \sim k^2$$

Wigner distribution

$$\widehat{W}_\epsilon(\eta, k, t) = \langle \psi^*(k - \epsilon\eta, t)\psi(k + \epsilon\eta, t)\rangle$$

$$W_\epsilon(y, k, t) = \epsilon \int_{(\mathbb{T}/\epsilon)} e^{i2\pi y\eta}\widehat{W}_\epsilon(\eta, k, t)\, d\eta$$

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Start with initial distribution such that $\sup_\epsilon \epsilon < \|\phi(0)\|^2 > \leq C$, and

$$\bar{W}_\epsilon(\eta, 0) = \int \widehat{W}_\epsilon(\eta, k, 0)\, dk \longrightarrow W_0(\eta)$$

Theorem

$$\widehat{W}_\epsilon(\eta, k, \epsilon^{-3/2}t) \underset{\epsilon\to 0}{\longrightarrow} e^{-\hat{c}t|\eta|^{3/2}} W_0(\eta)$$

with $\hat{c} = \sqrt{\frac{2}{3\gamma^5}}\hat{\alpha}''(0)^{3/4}$.

previous results

- *Basile, O. Spohn, ARMA 2009:* kinetic limit/weak noise: $\gamma = \epsilon\gamma'$, $\delta = 1$, convergence to a linear Boltzmann equation

$$\partial_t \widehat{W}(\eta, k, t) + i\eta\omega'(k) W(\eta, k, t) = \mathcal{L}\widehat{W}(\eta, k, t)$$

with the scattering operator

$$\mathcal{L}\widehat{W}(\eta, k, t) = \int R(k, k')(\widehat{W}(\eta, k', t) - \widehat{W}(\eta, k, t))$$

$$R(k, k') = R(k', k) = r^2(k, k - k') + r^2(k, k + k')$$

$$= \frac{3}{4} \sum_{i=\pm 1} R_i(k) R_{-i}(k')$$

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- *Jara, Komorowski, O. APP 2009*: invariance principle for the corresponding Markov process:

$$\lambda X(\lambda^{-3/2} t) \;=\; \lambda \int_0^{\lambda^{-3/2} t} \omega'(K(s))\, ds \underset{\lambda \to 0}{\longrightarrow} \frac{3}{2} - \mathit{stable\ Levy}$$

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- *Milton Jara, 2013*: 2 conserved quantity model

Time evolution of the Wigner distribution

Anti-Wigner distribution:

$$\widehat{Z}_\epsilon(\eta, k, t) = \langle \psi(k - \epsilon\eta, t)\psi(k + \epsilon\eta, t) \rangle$$

for small ϵ, $\delta = 2/3$,

$$\begin{aligned}
\partial_t \widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) &= -i\epsilon^{-\delta+1}\omega'(k)\eta\partial_t \widehat{W}_\epsilon + \gamma\epsilon^{-\delta}\mathcal{L}\widehat{W}_\epsilon \\
&\quad - \gamma\epsilon^{-\delta}\mathcal{L}\left[\widehat{Z}_\epsilon(\eta, k, t) + \widehat{Z}_\epsilon(-\eta, k, t)\right] \\
&\quad - \gamma\epsilon^{2-\delta}\eta^2\widetilde{\mathcal{L}}\,\widehat{W}_\epsilon + o(\epsilon) \\
\partial_t \widehat{Z}_\epsilon(\eta, k, \epsilon^{-\delta} t) &= \dots
\end{aligned}$$

Time evolution of the Wigner distribution

Anti-Wigner distribution:

$$\widehat{Z}_\epsilon(\eta, k, t) = \langle \psi(k - \epsilon\eta, t)\psi(k + \epsilon\eta, t)\rangle$$

for small ϵ, $\delta = 2/3$,

$$\begin{aligned}
\partial_t \widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) = & -i\epsilon^{-\delta+1}\omega'(k)\eta\partial_t \widehat{W}_\epsilon + \gamma\epsilon^{-\delta}\mathcal{L}\widehat{W}_\epsilon \\
& - \gamma\epsilon^{-\delta}\mathcal{L}\left[\widehat{Z}_\epsilon(\eta, k, t) + \widehat{Z}_\epsilon(-\eta, k, t)\right] \\
& - \gamma\epsilon^{2-\delta}\eta^2\widetilde{\mathcal{L}}\,\widehat{W}_\epsilon + o(\epsilon) \\
\partial_t \widehat{Z}_\epsilon(\eta, k, \epsilon^{-\delta} t) = & \dots
\end{aligned}$$

- Because of fast time fluctuations $\widehat{Z}_\epsilon(\eta, k, \epsilon^{-\delta} t) \longrightarrow 0$.
- By averaging for many collisions $\widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) \longrightarrow \bar{W}(\eta, t)$, constant in k.

Let's drop $\widehat{Z}_\epsilon = 0$ and smaller terms in ϵ, we are left

$$\partial_t \widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) = -i\epsilon^{-\delta+1}\omega'(k)\eta\partial_t \widehat{W}_\epsilon + \gamma\epsilon^{-\delta}\mathcal{L}\widehat{W}_\epsilon$$

$$\mathcal{L}\widehat{W}_\epsilon(\eta, k, t) = \int R(k, k')(\widehat{W}(\eta, k', t) - \widehat{W}_\epsilon(\eta, k, t))dk'$$

$$R(k, k') = \frac{3}{4}\sum_{i=\pm 1} R_i(k)R_{-i}(k')$$

To simplify the argument assume the simpler scattering rate

$$R(k, k') = R(k)R(k'), \qquad \int R(k)dk = 1, \qquad R(k) \sim k^2$$

$$\mathcal{L}\widehat{W}_\epsilon(\eta, k, t) = R(k)\int R(k')\widehat{W}_\epsilon(\eta, k', t)dk - R(k)\widehat{W}_\epsilon(\eta, k, t)$$

$$\equiv R(k) < R, \widehat{W}_\epsilon > - R(k)\widehat{W}_\epsilon(\eta, k, t)$$

Taking Laplace Transform $\hat{w}_\epsilon(\eta, k, \lambda) = \int_0^\infty e^{-\lambda t} \widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) dt$, and denoting $< R, \hat{w}_\epsilon >= \int R(k') \hat{w}_\epsilon(\eta, k', \lambda) dk'$

$$\epsilon^\delta \lambda \hat{w}_\epsilon + i\epsilon\omega'(k)\eta \hat{w}_\epsilon - R(k) < R, \hat{w}_\epsilon > + R(k) \hat{w}_\epsilon = \epsilon^\delta W_0(\eta, k)$$

Taking Laplace Transform $\hat{w}_\epsilon(\eta, k, \lambda) = \int_0^\infty e^{-\lambda t}\widehat{W}_\epsilon(\eta, k, \epsilon^{-\delta} t) dt$, and denoting $< R, \hat{w}_\epsilon >= \int R(k')\hat{w}_\epsilon(\eta, k', \lambda) dk'$

$$\epsilon^\delta \lambda \hat{w}_\epsilon + i\epsilon\omega'(k)\eta\hat{w}_\epsilon - R(k) < R, \hat{w}_\epsilon > + R(k)\hat{w}_\epsilon = \epsilon^\delta W_0(\eta, k)$$

rearranging

$$w_\epsilon(\eta, k, \lambda) = \frac{\epsilon^\delta W_0(\eta, k) + R(k) < R, \hat{w}_\epsilon >}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta}$$

Multiplying by $R(k)$ and integrating in k, after rearrangement:

$$< R, \hat{w}_\epsilon > \epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right) = \int \frac{R(k) W_0(\eta, k)}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk$$

$$< R, \hat{w}_\epsilon > \epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right)$$

$$= \int \frac{R(k)\widehat{W_0}(\eta, k)}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk$$

$$< R, \hat{w}_\epsilon > \epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta}dk\right)$$
$$= \int \frac{R(k)\widehat{W_0}(\eta, k)}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta}dk$$

$$\int \frac{R(k)W_0(\eta, k)}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta}dk \longrightarrow \int \widehat{W_0}(\eta, k)dk = \bar{W}_0(\eta)$$

$$< R, \hat{w}_\epsilon > \epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right)$$

$$= \int \frac{R(k)\widehat{W}_0(\eta, k)}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk$$

$$\int \frac{R(k) W_0(\eta, k)}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk \longrightarrow \int \widehat{W}_0(\eta, k) dk = \bar{W}_0(\eta)$$

$$\epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right) \longrightarrow \lambda + \hat{c}\eta^{3/2}$$

$$< R, \hat{w}_\epsilon > \epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right)$$

$$= \int \frac{R(k)\widehat{W}_0(\eta, k)}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta} dk$$

$$\int \frac{R(k)W_0(\eta, k)}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta} dk \longrightarrow \int \widehat{W}_0(\eta, k) dk = \bar{W}_0(\eta)$$

$$\epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta\lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right) \longrightarrow \lambda + \hat{c}\eta^{3/2}$$

$$< R, \hat{w}_\epsilon >= \int R(k)\hat{w}_\epsilon(\eta, k, \lambda) dk \longrightarrow w(\eta, \lambda).$$

$$< R, \hat{w}_\epsilon > \epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right)$$

$$= \int \frac{R(k)\widehat{W_0}(\eta, k)}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk$$

$$\int \frac{R(k)W_0(\eta, k)}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk \longrightarrow \int \widehat{W_0}(\eta, k) dk = \bar{W}_0(\eta)$$

$$\epsilon^{-\delta}\left(1 - \int \frac{R(k)^2}{\epsilon^\delta \lambda + R(k) + i\epsilon\omega'(k)\eta} dk\right) \longrightarrow \lambda + \hat{c}\eta^{3/2}$$

$$< R, \hat{w}_\epsilon >= \int R(k)\hat{w}_\epsilon(\eta, k, \lambda) dk \longrightarrow w(\eta, \lambda).$$

$$\Longrightarrow \quad \left(\lambda + \hat{c}\eta^{3/2}\right) w(\eta, \lambda) = \bar{W}_0(\eta) \qquad \square$$