The Matching Problem is in Quasi-NC

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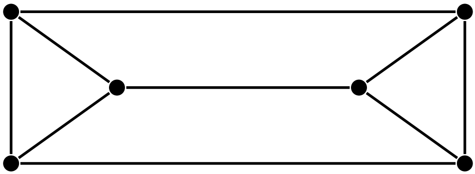
Perfect matching problem

Given a graph, can we pair up all vertices using edges?



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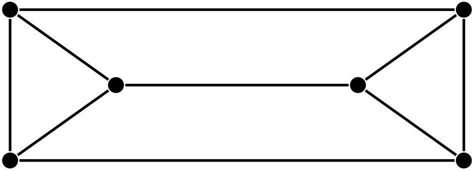
graph is non-bipartite!



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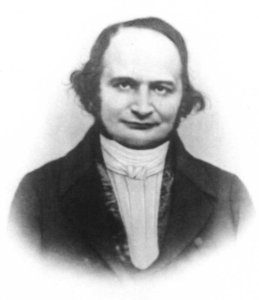
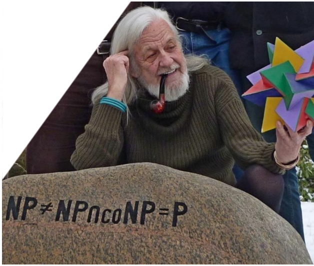
Benchmark problem in computer science





Perfect matching problem

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Algorithms:

- bipartite: Jacobi [XIX century, weighted!]
- general: Edmonds [1965]
  - polynomial-time = efficient
- since then, tons of research
  and still active
- many models of computation:
  monotone circuits, extended formulations,
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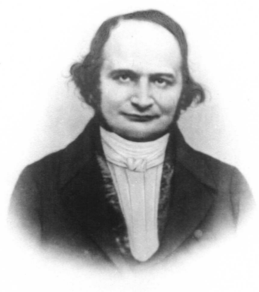

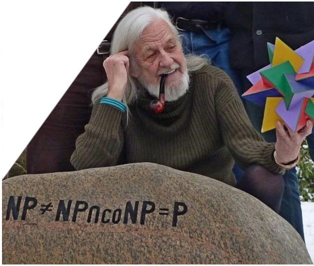

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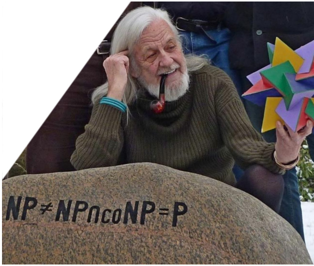

Parallel complexity

Class $\mathcal{NC}$: problems that parallelize completely

poly n processors

poly log n time

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It's in Randomized $\mathcal{NC}$

poly log n time

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Matching is in RANDOMIZED $\mathcal{NC}$ [Lovász 1979]:
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Can we derandomize efficient computation?

Can we derandomize one of these algorithms?

Is matching in $\mathcal{NC}$?

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Yes, for restricted graph classes:

- bipartite regular [Lev, Pippenger, Valiant 1981]
- bipartite convex [Dekel, Sahni 1984]
- incomparability graphs [Kozen, Vazirani, Vazirani 1985]
- bipartite graphs with small number of perfect matchings [Grigoriev, Karpinski 1987]
- claw-free [Chrobak, Naor, Novick 1989]
- $K_{3,3}$-free (decision version) [Vazirani 1989]
- planar bipartite [Miller, Naor 1989]
- dense [Dahlhaus, Hajnal, Karpinski 1993]
- strongly chordal [Dahlhaus, Karpinski 1998]
- P_4-tidy [Parfenoff 1998]
- bipartite small genus [Mahajan, Varadarajan 2000]
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- planar (search version) [Anari, Vazirani 2017]

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but not known for:

- **bipartite**

Theorem Fenner, Gurjar and Thierauf [2015]

Bipartite matching is in $\textsc{quasi-}\mathcal{NC}$

($n^{\text{poly} \log n}$ processors, poly log n time, deterministic)



Theorem Fenner, Gurjar and Thierauf [2015]

Bipartite matching is in QUASI-$\mathcal{NC}$

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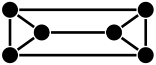
- Approach fails for non-bipartite graphs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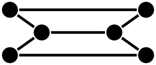

Theorem S. and Tarnawski [2017]

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with quasi-polynomial
processors

Outline

1. Basic approach for derandomization
2. Bipartite case [Fenner, Gurjar, Thierauf 2015]
3. Difficulties of general case & our approach

Basic approach for derandomization

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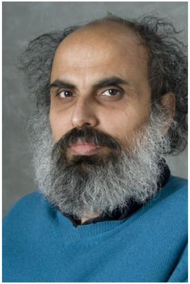
(Derandomize one of the randomized algorithms)

Algorithm of Mulmuley, Vazirani, Vazirani'87



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something deterministic?

Construct isolating w in $\mathcal{NC}$?

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Easy even with $|\mathcal{W}^*| \leq 1$

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Easy, but best known bound on $|\mathcal{W}^*|$ is exponential in n

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Thm[FGT'15]: $\mathcal{W}^*$ exists for bipartite graphs

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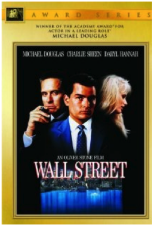
Thm[FGT'15]: $\mathcal{W}^*$ exists for bipartite graphs

Thm[ST'17]: $\mathcal{W}^*$ exists for general graphs

Bipartite case

[Fenner, Gurjar, Thierauf 2015]



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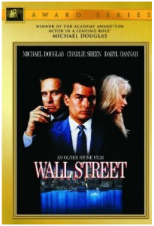

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Construct isolating function iteratively

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Let $\mathcal{W} = \{w_k : w_k(e_i) = 2^i \mod k \text{ for } k = 2, 3, \ldots, n^4\}$ be a polynomial set of simple weight functions

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All matchings of G

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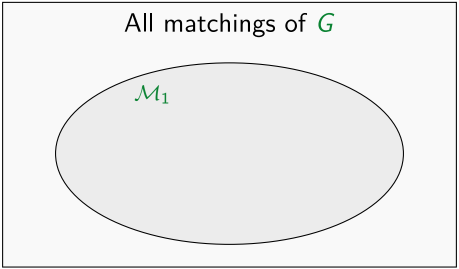

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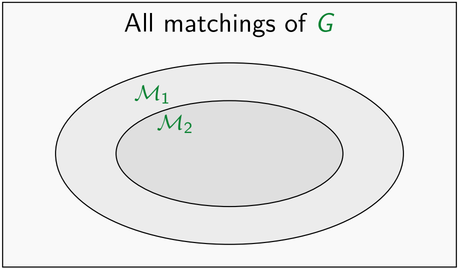

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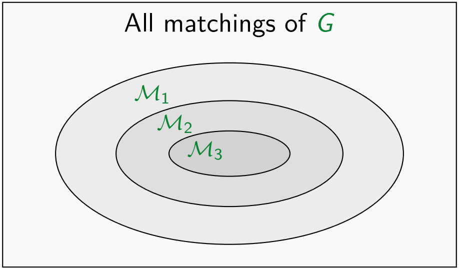

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How many $w_1, \ldots, w_\ell \in \mathcal{W}$ necessary for $|\mathcal{M}_\ell| = 1$?

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$$\Downarrow$$

$$\mathcal{W}^* = \{n^{9(\log(n))} w_1 + n^{9(\log(n)-1)} w_2 + \cdots + 1 \cdot w_{\log(n)} : w_1, \ldots, w_{\log_2(n)} \in \mathcal{W}\}$$

gives oblivious quasi-polynomial derandomization

GOAL: For *any* n-vertex graph G, show that there is

$$w_1, \ldots, w_{\log n} \in \mathcal{W} = \{w_k : w_k(e_i) = 2^i \mod k \text{ for } k = 2, 3, \ldots, n^4\}$$

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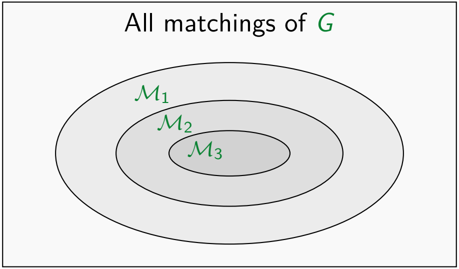

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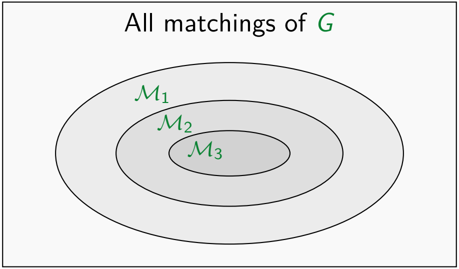

We need good progress measure

Minimum perfect matchings of the same weight

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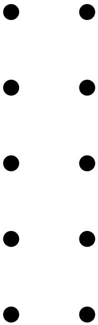

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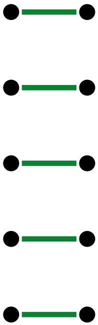

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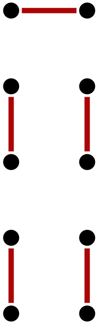

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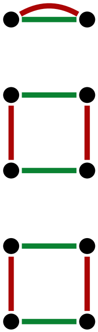

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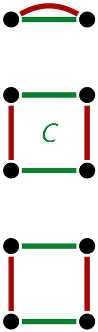

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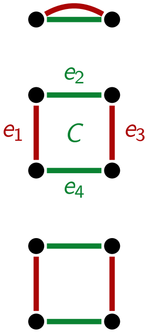

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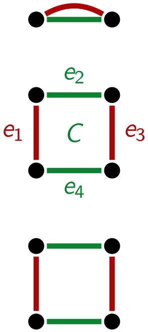

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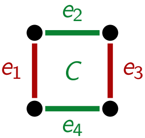

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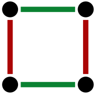

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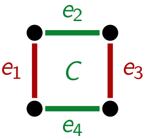

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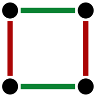

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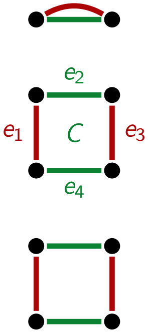

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Progress: assign $\neq 0$ discrepancy to “many” cycles

Removing cycles

A graph may have exponentially many cycles $\Rightarrow$ seems hard to find w so that all of them have non-zero discrepancy

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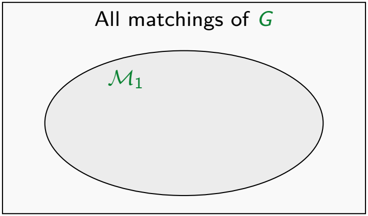
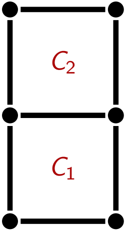
Not so easy, but we can cope with all **4-cycles**

Select $w_1 \in \mathcal{W}$ so that all 4-cycles have $\neq 0$ discrepancy





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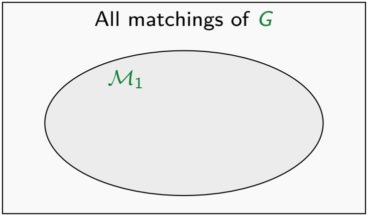

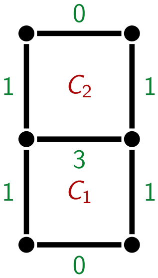

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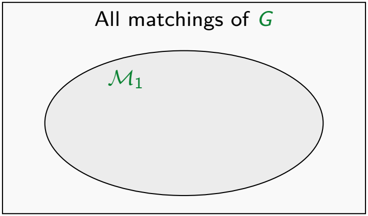

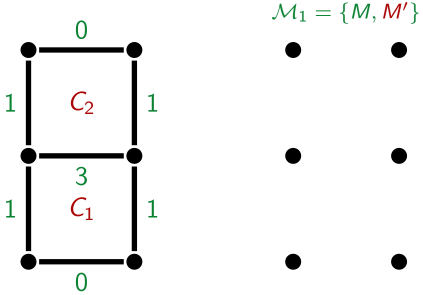

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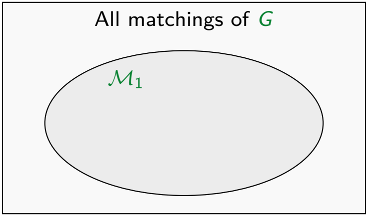
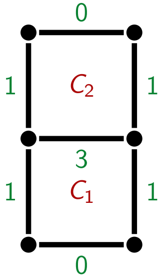
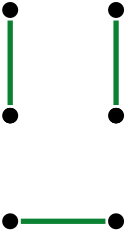




$\mathcal{M}_1 = \{M, M'\}$



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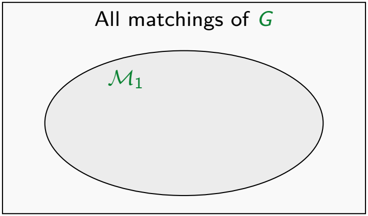

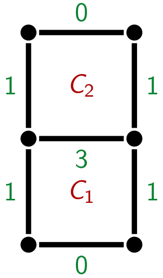

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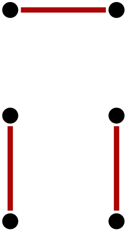

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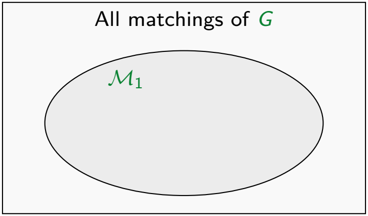

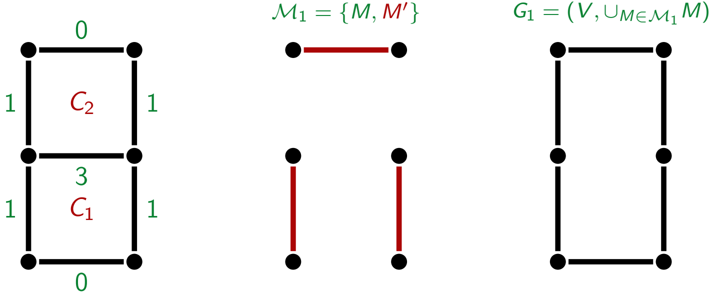

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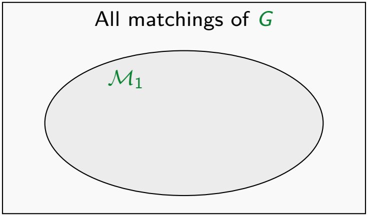

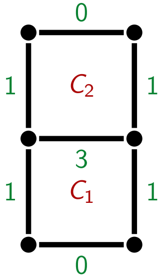

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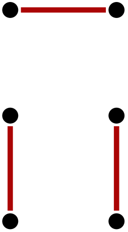
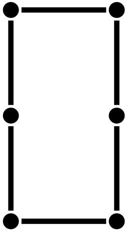


$G_1 = (V, \cup_{M \in \mathcal{M}_1} M)$



What can we say about the active subgraph G_1 that contains those edges that are in a min-weight perfect matching?

Bipartite key property: Once we assign a cycle $\neq 0$ discrepancy, it will disappear from the active subgraph

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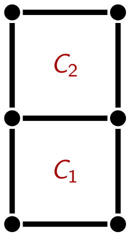

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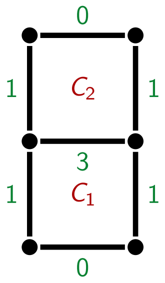

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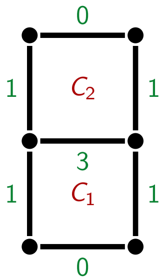

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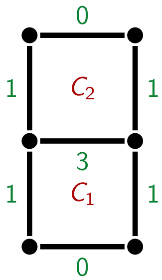

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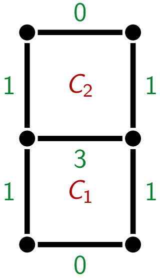

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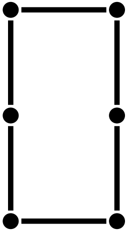

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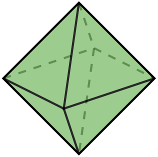
Proof: Let $\mathcal{M}$ be the set of perfect matchings minimizing w

- Consider the convex hull of $\mathcal{M}$ (face F of the bipartite matching polytope):



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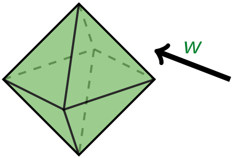

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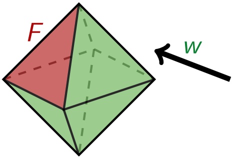

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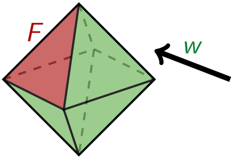

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Bipartite PM

$$\begin{aligned} x(\delta(v)) &= 1 && \text{for every } v \in V \\ x_e &\geq 0 && \text{for every } e \in E \end{aligned}$$

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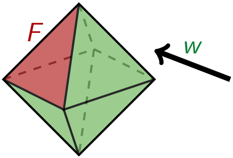

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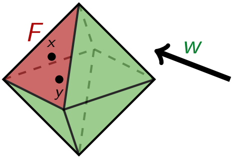

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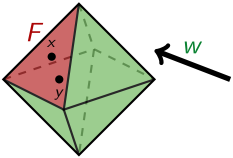

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Bipartite PM

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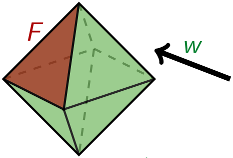
$x_e \geq 0$	for every $e \in E$

F is simply a subgraph

- What can we say about the weight of points in F?

  Every $x, y \in F$ have same weight: $\sum_e w(e)x_e = \sum_e w(e)y_e$

F is the convex hull of $\mathcal{M} \Rightarrow$ every $x, y \in F$ have same weight



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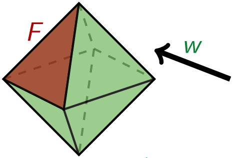

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(edge set $\cup_{M \in \mathcal{M}} M$)

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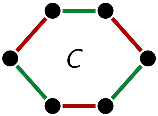

w(green edges) $\neq$ w(red edges)

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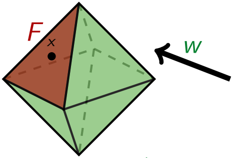

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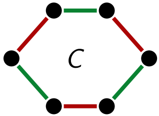
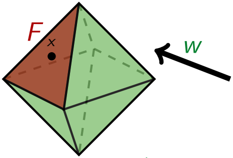


$w(\text{green edges}) \neq w(\text{red edges})$

- Let $x = \frac{1}{|\mathcal{M}|} \sum_{M \in \mathcal{M}} \mathbf{1}_M$ be the mean of the face F

F is the convex hull of $\mathcal{M}$ $\Rightarrow$ every $x, y \in F$ have same weight



PM : perfect matching polytope (convex hull of matchings)

Bipartite PM

$$x(\delta(v)) = 1 \qquad \text{for every } v \in V$$
$$x_e \geq 0 \qquad \text{for every } e \in E$$

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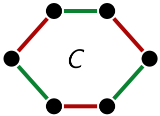

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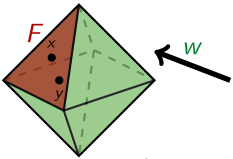

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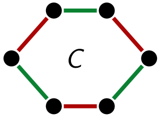
(edge set $\cup_{M \in \mathcal{M}} M$)

- Suppose active subgraph has cycle C of $\neq 0$ discepancy



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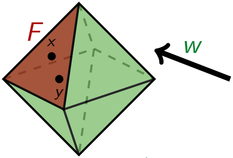

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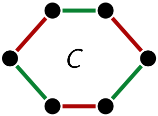

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- Increasing red edges while decreasing green maintain degrees
- So we obtain a new point $y \in F$ of different weight; contradiction

The main ingredients

Old Lemma:

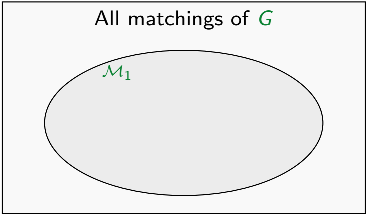
For any collection of n^4 cycles, **some** $w \in \mathcal{W}$ assigns **all of them** $\neq 0$ discrepancy

Bipartite key property:

Once we assign a cycle $\neq 0$ discrepancy,
it will disappear from the active subgraph

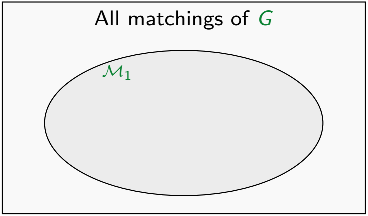
Select $w_1 \in \mathcal{W}$ so that all 4-cycles in G have $\neq 0$ discrepancy

A graph has at most n^4 cycles of length 4



Select $w_1 \in \mathcal{W}$ so that all 4-cycles in G have $\neq 0$ discrepancy

- Bipartite key property: $G_1 = (V, \cup_{M \in \mathcal{M}_1} M)$ has no cycles of length ≤ 4



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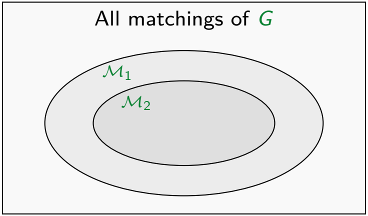
Select $w_2 \in \mathcal{W}$ so that all ≤ 8-cycles in G_1 have $\neq 0$ discrepancy

A graph with no ≤ 4-cycles has at most n^4 cycles of length ≤ 8



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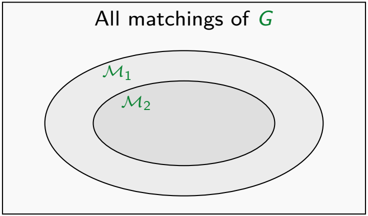

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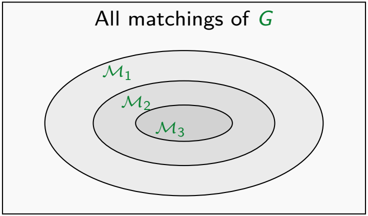
A graph with no ≤ 8-cycles has at most n^4 cycles of length ≤ 16



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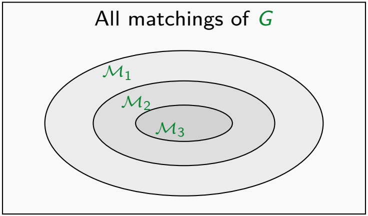
$$\vdots$$



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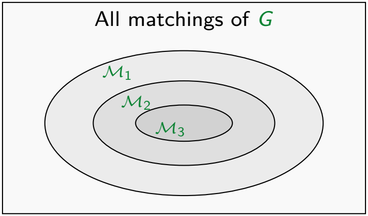
$$\vdots$$

$G_{\log n} = (V, \cup_{M \in \mathcal{M}_{\log n}} M)$ have no cycles so $|\mathcal{M}_{\log n}| = 1$ as required



Final argument

A graph with no ≤ 4-cycles has at most n^4 cycles of length 8

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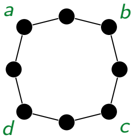
A graph with no ≤ 4-cycles has at most n^4 cycles of length 8

- Associate a signature (a, b, c, d) with each 8-cycle
  - a is the first vertex, b is the third vertex, c is the fifth vertex, d is the seventh vertex



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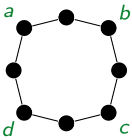

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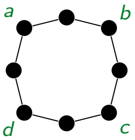

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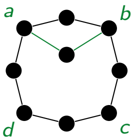

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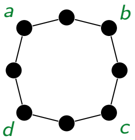

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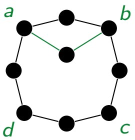
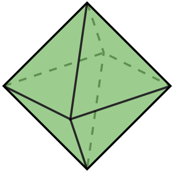


- So # 8-cycles is at most # signatures which is at most n^4

Some perspective

Polyhedral perspective

1



isolating in stages
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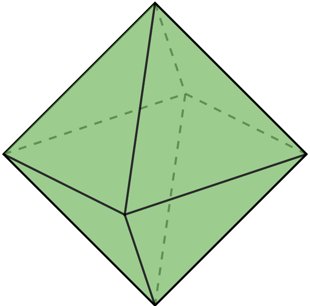
decreasing sequence of faces



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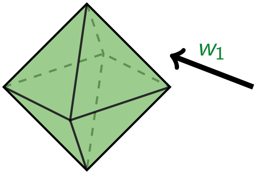

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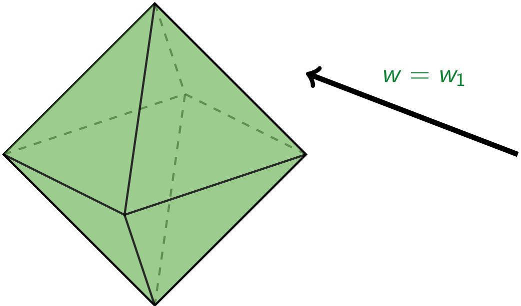

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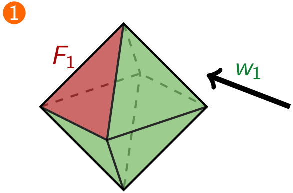

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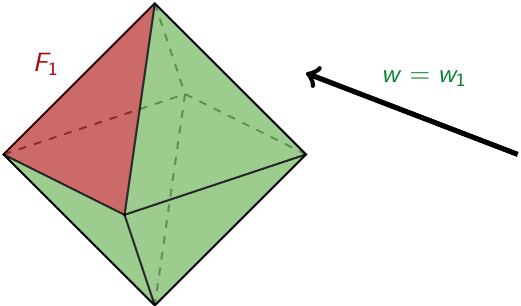

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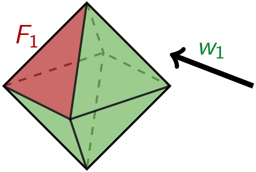

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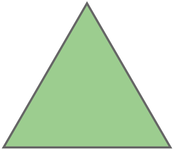

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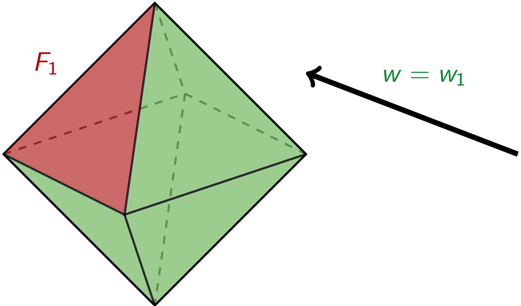

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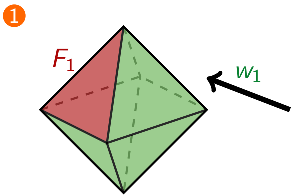

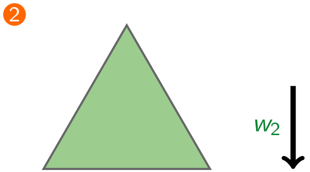

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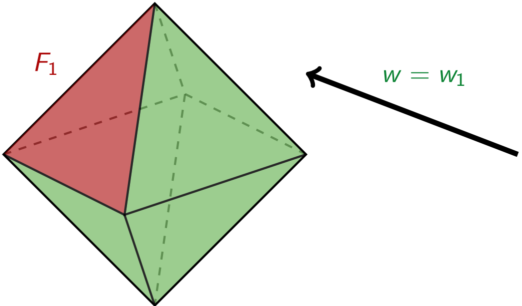

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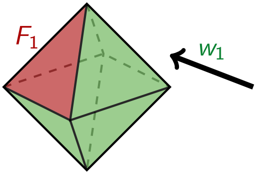

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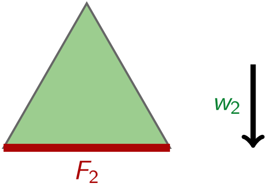

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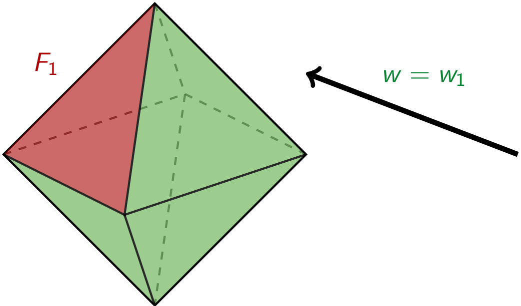

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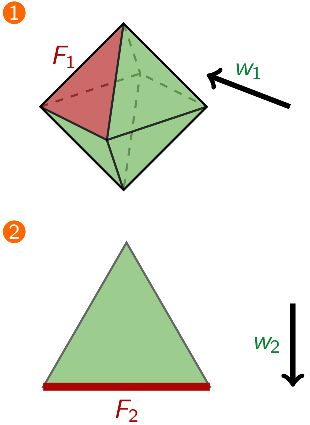

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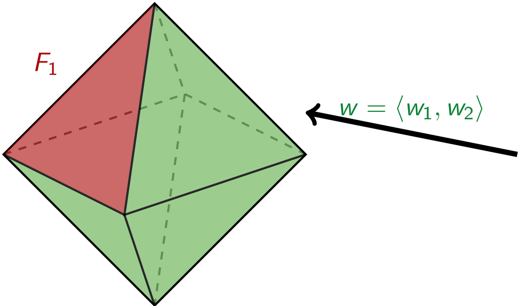

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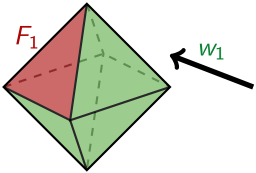

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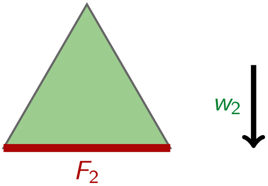

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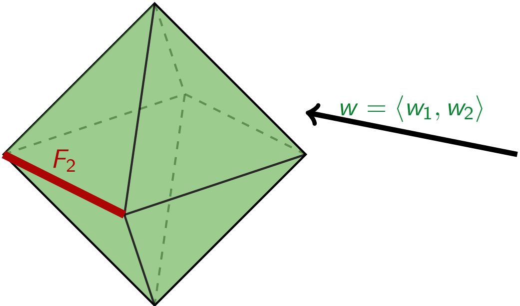

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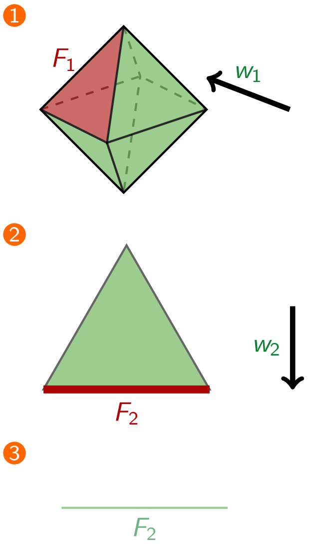

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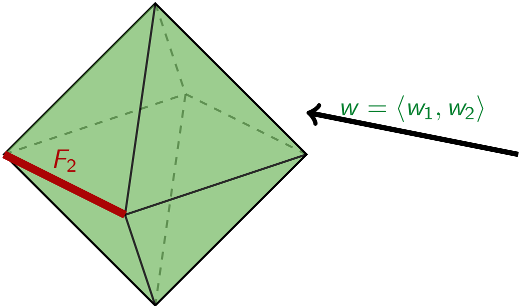

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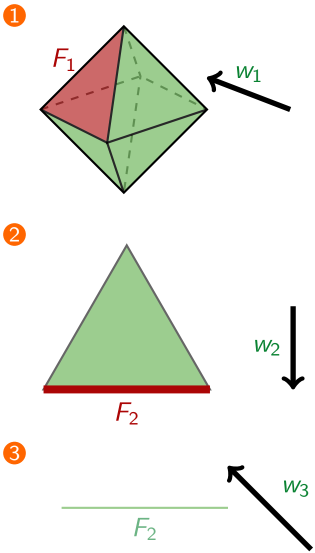

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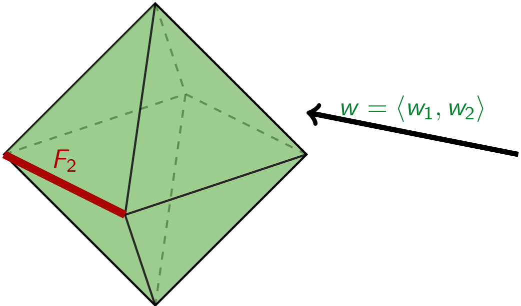

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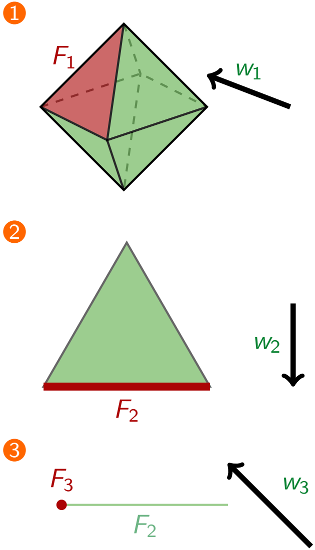

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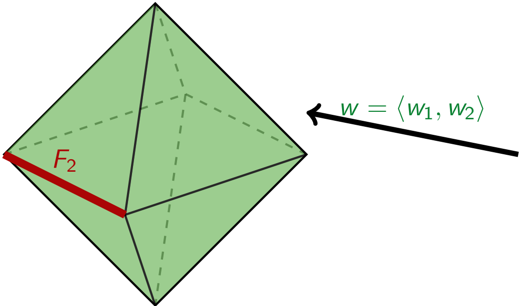

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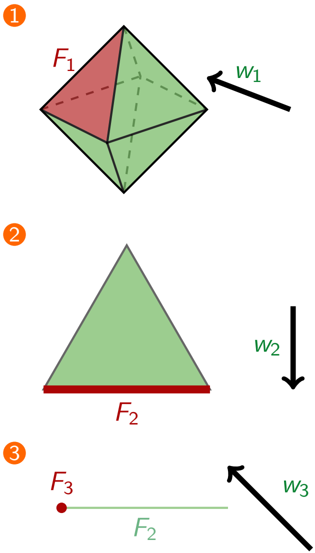

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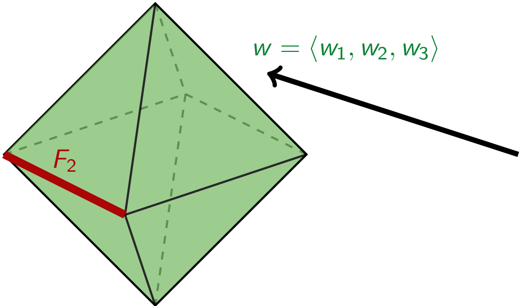

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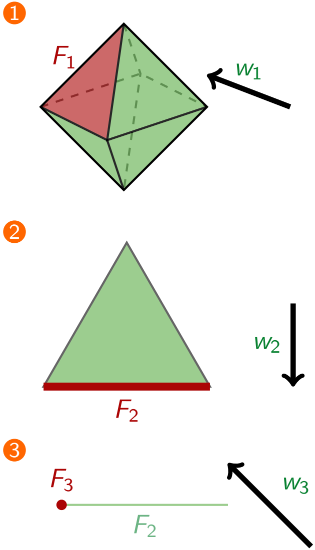

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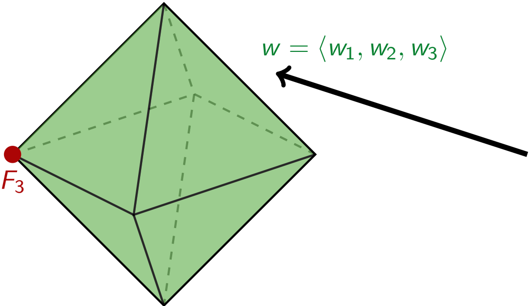

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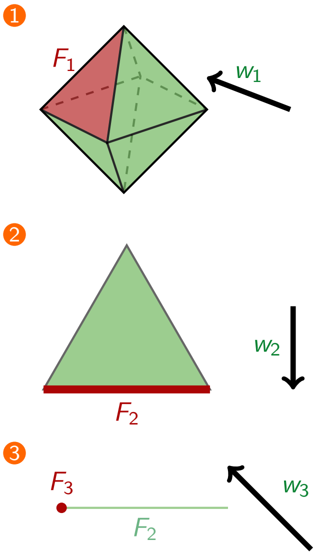

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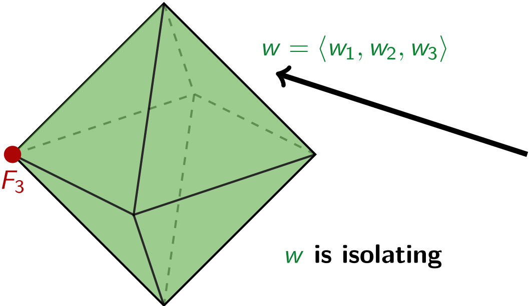

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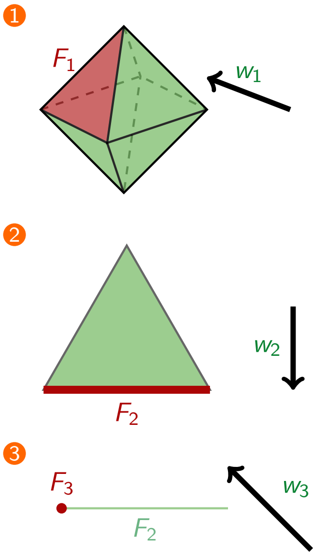
Polyhedral perspective



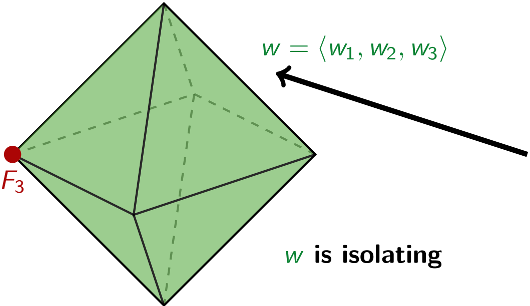
isolating in stages

=

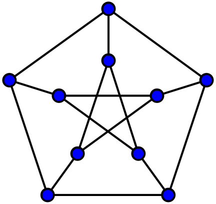
decreasing sequence of faces

Fast decrease due to bipartite matching polytope:

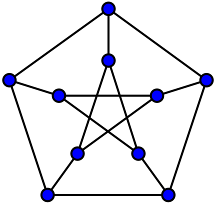
- every face is a subgraph
- Key property: girth doubles in every step



w **is isolating**



Difficulties of general case & our approach



Difficulties of general case & our approach

~~**Bipartite key property:** Once we assign a cycle $\neq 0$ discrepancy, it will disappear from the active subgraph~~

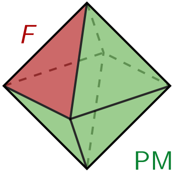
General graphs are "exponentially" harder

Edmonds [1965] Perfect matching polytope description on $x \in \mathbb{R}^E$:

- $x_e \geq 0$ for every edge e
- $x(\delta(v)) = 1$ for every vertex v

 $x(\delta(S)) \geq 1$ for every odd set S of vertices

($\delta(S)$ = edges crossing S)



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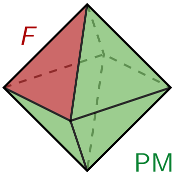

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  $F = \{x \in \text{PM} : \; x_e = 0 \;\; \text{for some edges } e\}$
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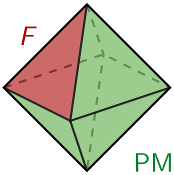
- Now, faces are exponentially harder
- Need $2^{\Omega(n)}$ inequalities [Rothvoss 2013]



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Girth does not make sense as progress measure and bipartite key property fails!

$x(\delta(S)) = 1$ for some odd sets $S\}$

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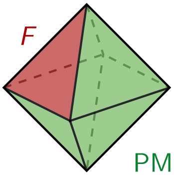
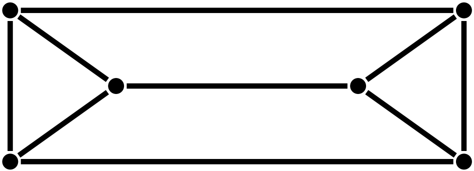


How bipartite key property fails



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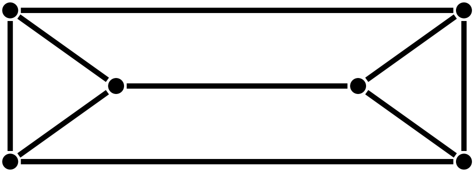
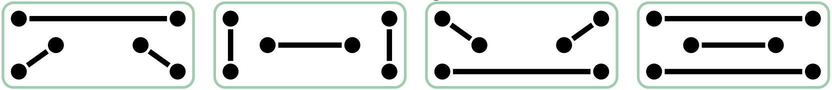


PM: convex hull of all four matchings:



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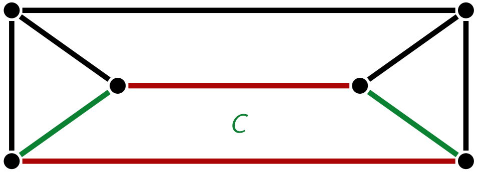

want:
$d_w(C) \neq 0$

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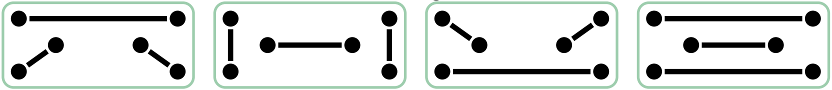

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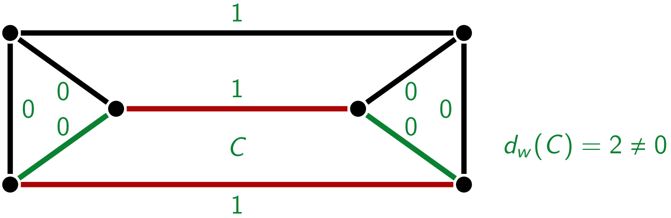

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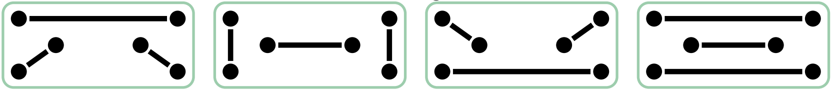

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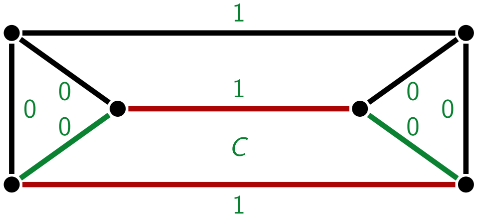

$d_w(C) = 2 \neq 0$

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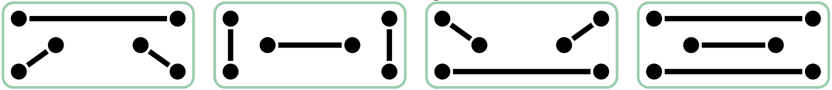

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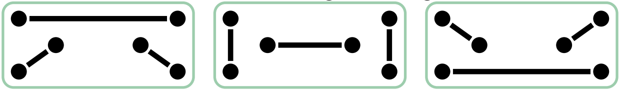

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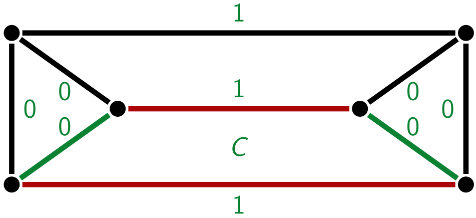

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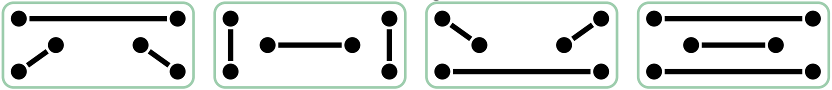

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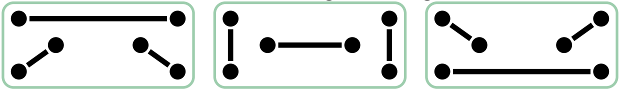

$F \subsetneq PM$ but still has all edges...

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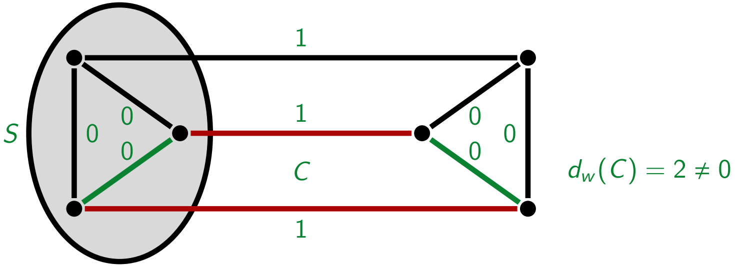

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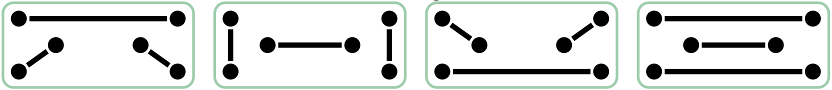

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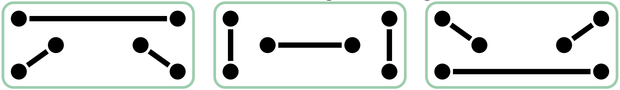

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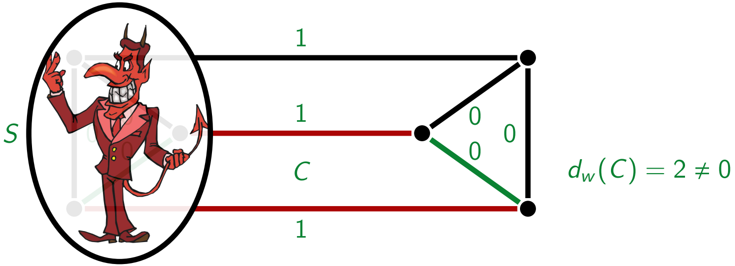

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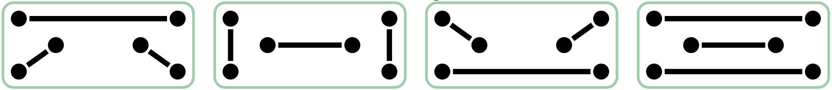

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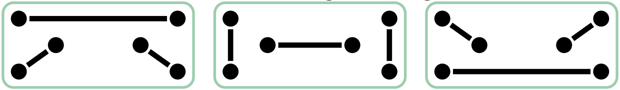

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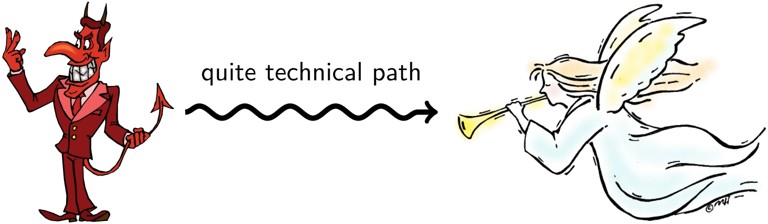
quite technical path



Main ingredients:

- Laminar family of tight constraints (at most $2n-1$ constraints instead of exponential)
- Tight cut constraints decompose the instance

  $\Rightarrow$ divide-and-conquer approach

Laminarity

Every face F is given as:

$$\begin{aligned} F = \{x \in \mathrm{PM} : x_e &= 0 && \text{for some edges } e, \\ x(\delta(S)) &= 1 && \text{for some odd sets } S\} \end{aligned}$$

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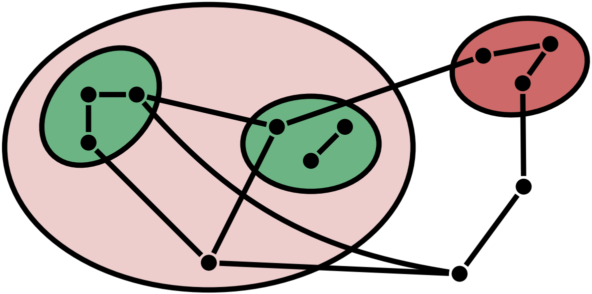
Great news: "some" can be chosen to be a laminar family!



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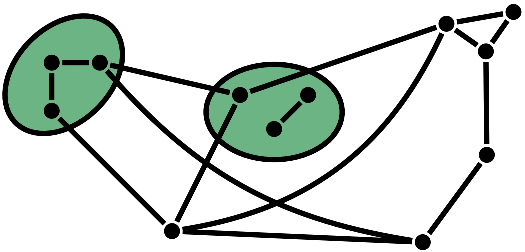

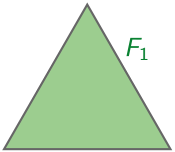

face $\sim$ (edge subset, laminar family)

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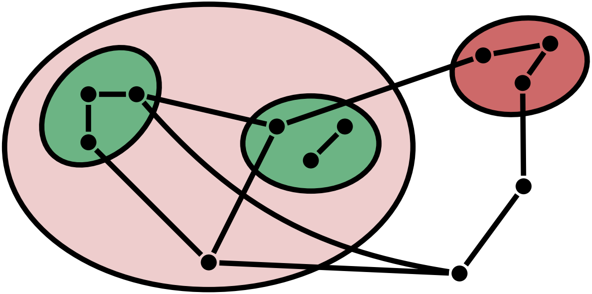

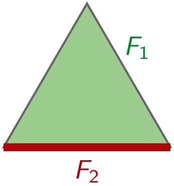

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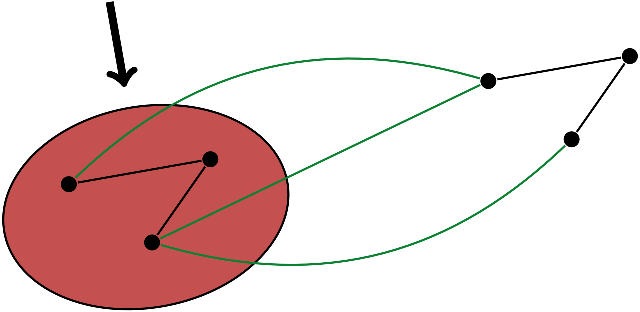
Tight odd cuts decomposes instance

exactly one edge crossing



- once we fix a boundary edge...

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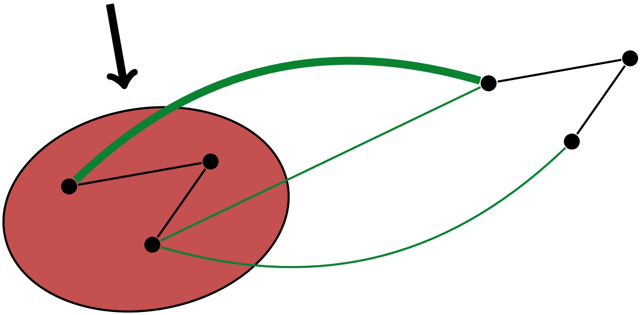

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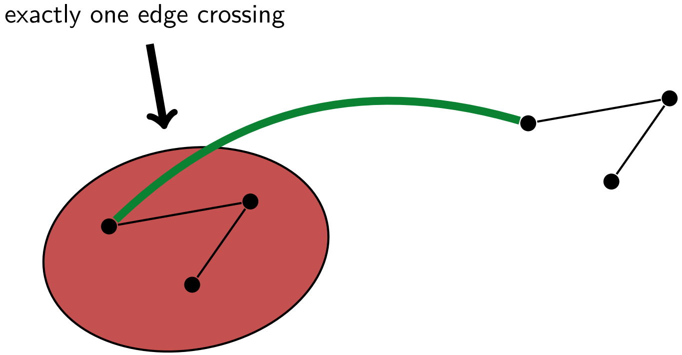

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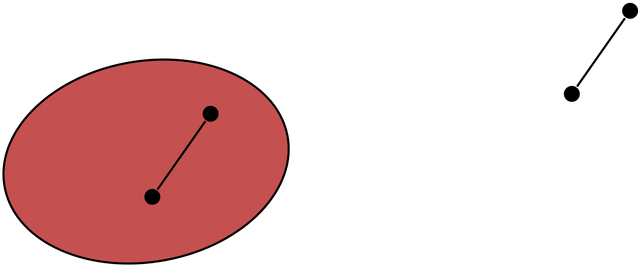

- once we fix a boundary edge...
- ... the instance decomposes into two **independent** ones

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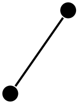

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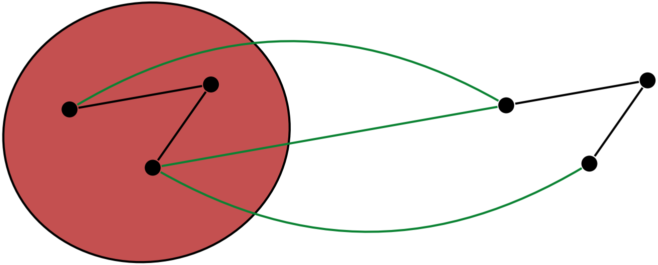
Divide & conquer

Between friends: cycles that do not cross tight odd sets behave like in the bipartite case and can thus be removed

Simplest case: only one tight odd set



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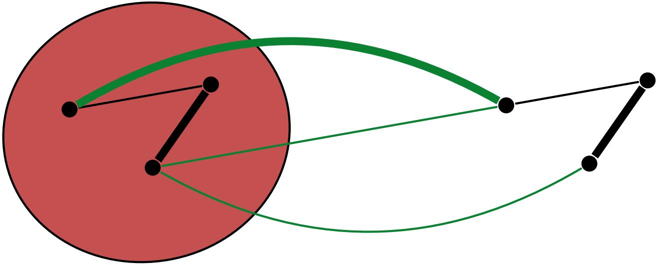

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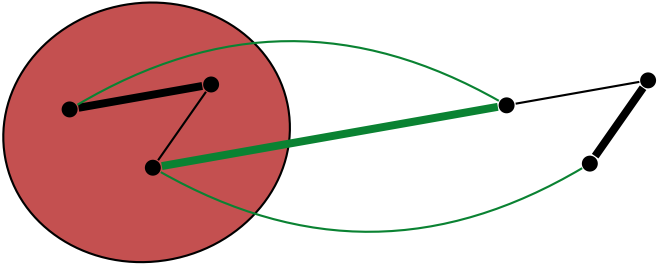

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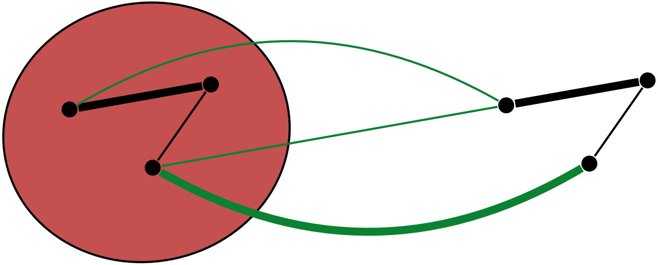

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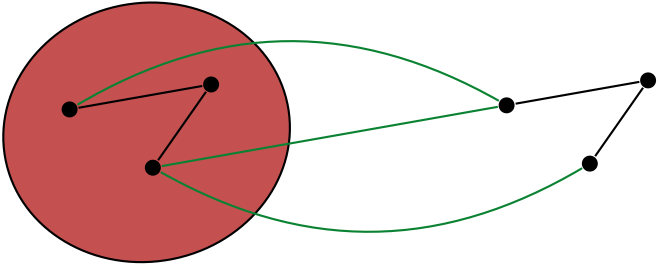

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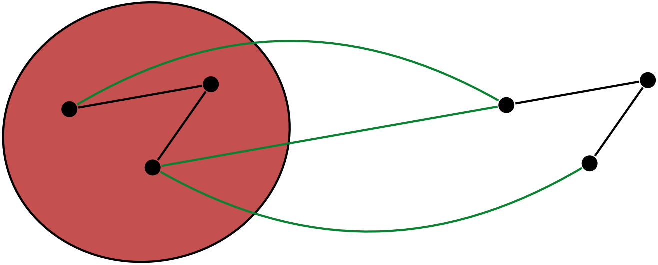
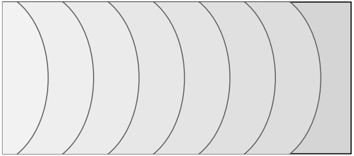


- then every boundary edge determines entire matching
- so: at most n^2 perfect matchings
- some $w \in \mathcal{W}$ will give them different weights

Divide & conquer: chain case



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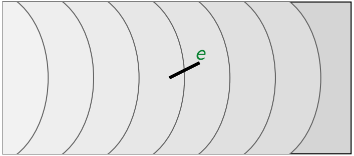

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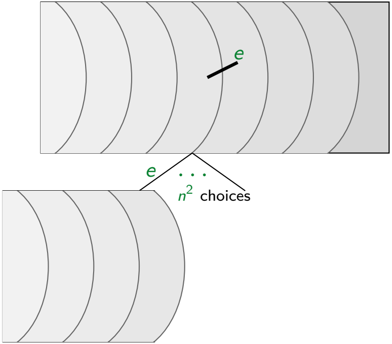

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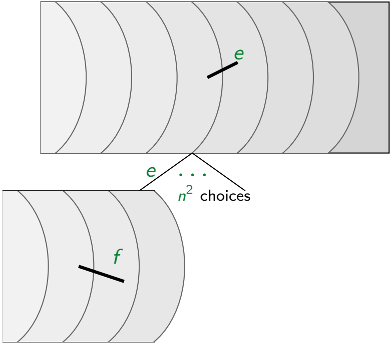

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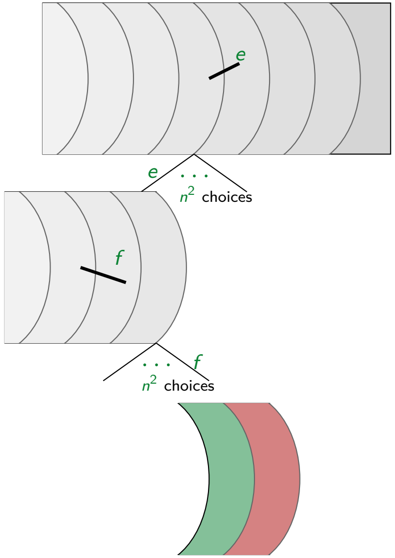

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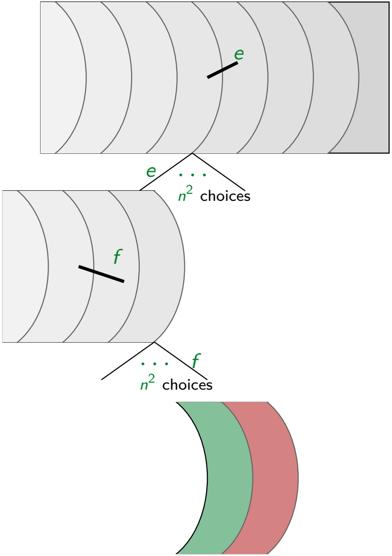
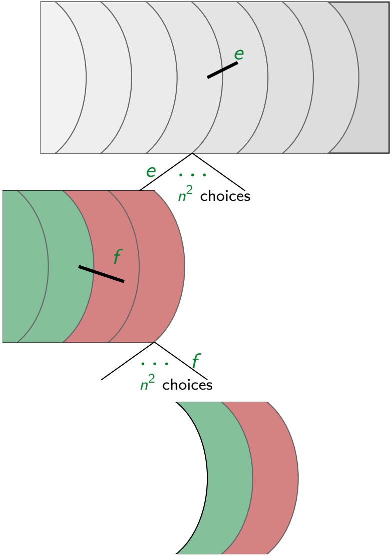


Instance where both sides of the cut are isolated,
one $w \in \mathcal{W}'$ makes the whole subinstance isolated

Divide & conquer: chain case



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Divide & conquer: chain case

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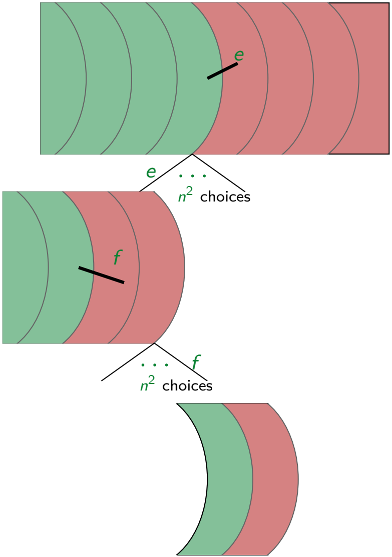

Divide & conquer: chain case

As before we isolate the whole instance in $O(\log n)$ phases

Now instance where both sides of the cut are isolated, one $w \in \mathcal{W}'$ makes the **whole instance isolated :)**

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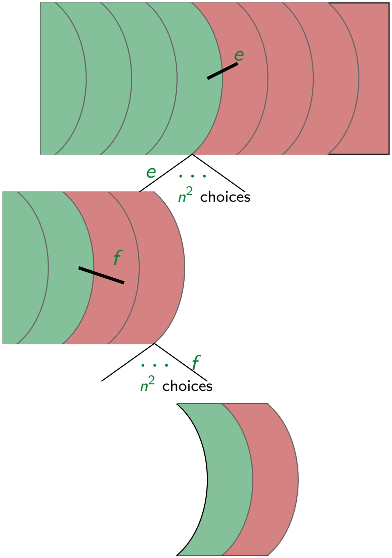
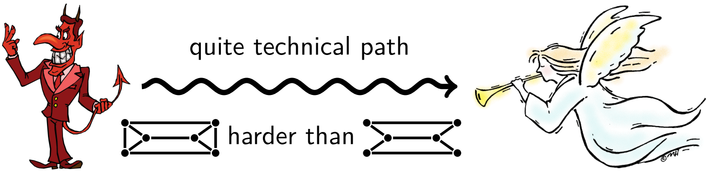
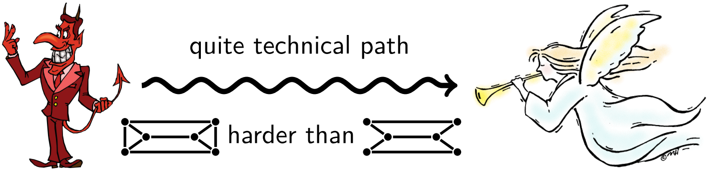
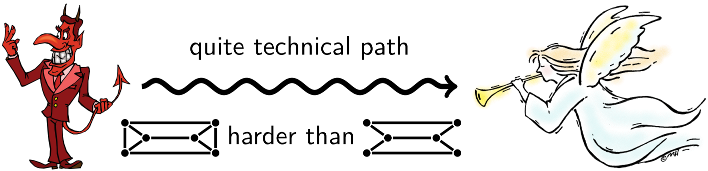



quite technical path
harder than



Carefully selected progress measure allows us to reduce laminar case to

- Removing cycles similar to bipartite case
- The chain case (divide-and-conquer)



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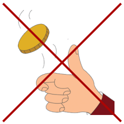
Theorem S. and Tarnawski [2017]

General matching is in QUASI-$\mathcal{NC}$



with quasi-polynomial # processors

Future work

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  - even for bipartite graphs
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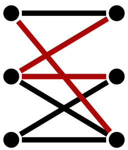
Exact Matching Problem



Given: graph with some edges red, number k.
Is there a perfect matching with exactly k red edges?

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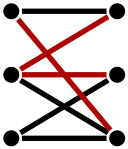

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Thank you!