Meridional essential surfaces of unbounded Euler characteristics in knot complements

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Introduction

Essential surfaces have a preeminent presence on the understanding of 3-manifold topology, and of knot complements in particular.

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Unbounded Euler characteristic

Lyon, 71:

- Knot complement with closed essential surfaces of arbitrarily high genus;
- Knot with incompressible Seifert surfaces of arbitrarily high genus.

Later, other examples with the same properties for closed surfaces were also obtained, for instance: Oertel, 84; Eudave-Muñoz and Neumann-Coto, 04; Y. Li, 09.

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> Can the arbitrarily large Euler characteristic property be obtained from the number of boundary components?
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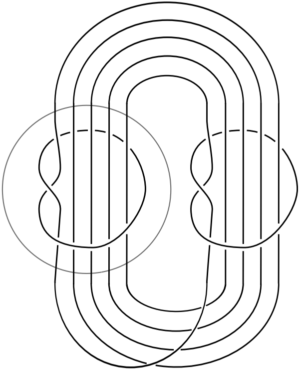
Arbitrarily high number of boundaries

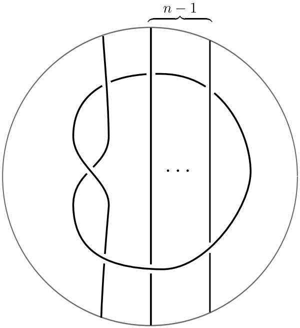
We start by considering meridional essential planar surfaces or, equivalently, essential tangle decompositions of knots.



n-string essential tangle

n-string tangle:
$(B, s_1 \cup \cdots \cup s_n)$, is a ball with n p.e. disjoint arcs.

Essential:
($n \geq 2$) if there is no disk, in $B - \cup_i s_i$, separating the collection of arcs;
(n=1) if the single string is knotted.



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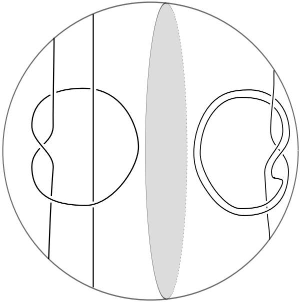

Not essential

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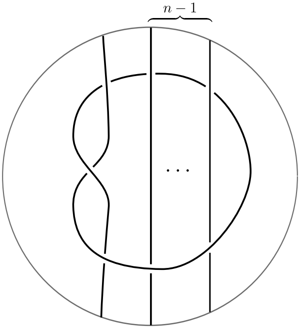
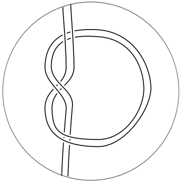
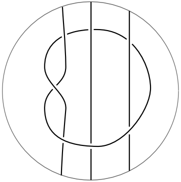




Essential



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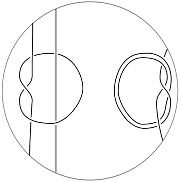
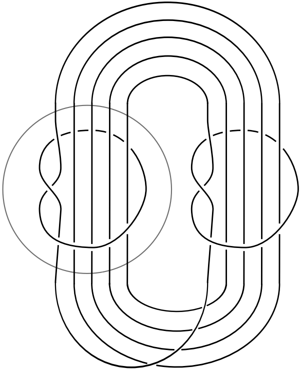


Not essential

n-string tangle decompositions

n-string tangle decomposition:
$(S^3, K) = (B_1, \mathcal{T}_1) \cup_S (B_2, \mathcal{T}_2)$
with $\mathcal{T}_i$ being n arcs.

Essential: if both $(B_i, \mathcal{T}_i)$ are essential.



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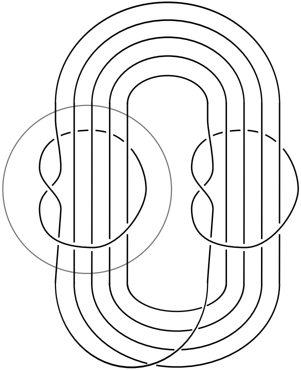

Knots and their essential tangle decompositions

Lickorish, 81: If a knot has a 2-string essential tangle decomposition with no local knots then the knot is prime.

Ozawa, 97: If a knot has a 2-string essential free tangle decomposition then this is the unique essential tangle decomposition of the knot up to isotopy.

Knots with no closed essential surfaces (CGLS, 87), tunnel number one knots (Gordon and Reid, 95), and free genus one knots (Matsuda and Ozawa, 98), have no essential tangle decompositions.

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Question:

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Essential tangle decompositions: arbitrarily high number of strings

Theorem 1 (N., 15).
There is an infinite collection of prime knots such that

for all $n \geq 2$

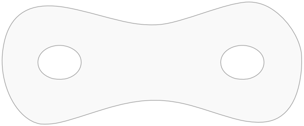
each knot has a n-string essential tangle decomposition.

Corollary.
There is an infinite collection of knots such that

for all $n \geq 1$

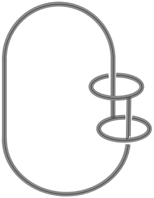
each knot has a n-string essential tangle decomposition.

Strategy for the proof



Handlebody of genus 2

$\longrightarrow$ embedding into S^3



Handlebody-knot 4_1

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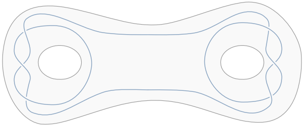

Connected sum of two trefoils

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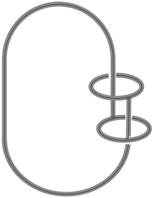

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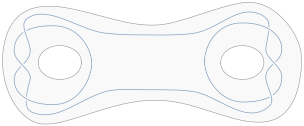

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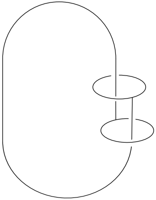
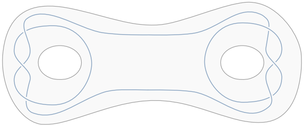


Spine of 4_1

Strategy for the proof



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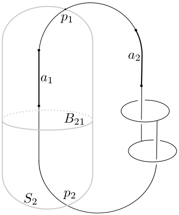

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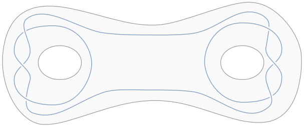

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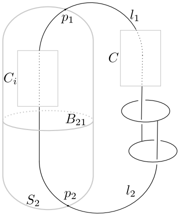

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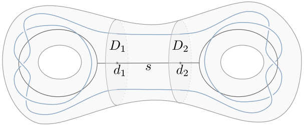

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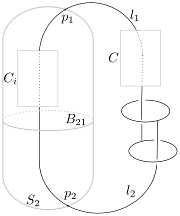

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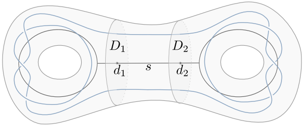

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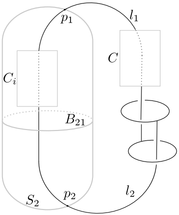

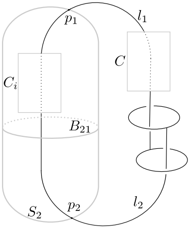

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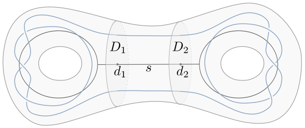

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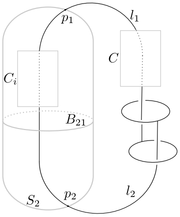

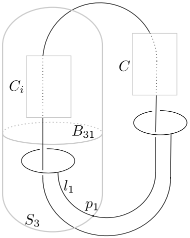

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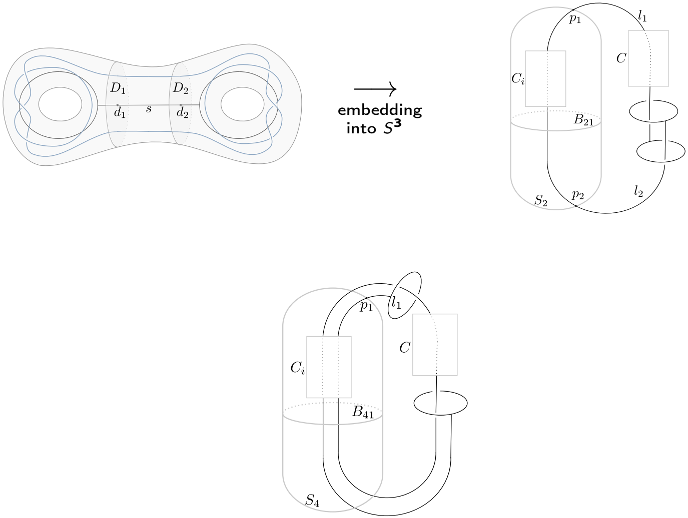

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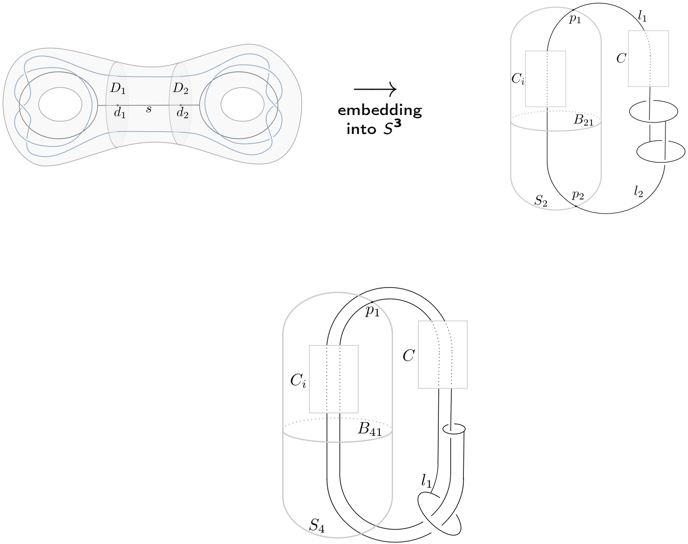

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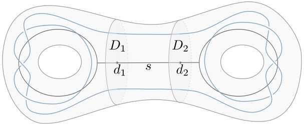

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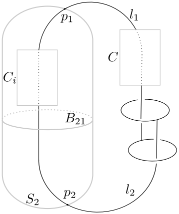

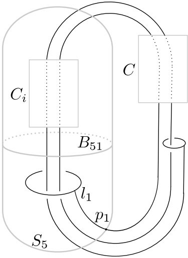

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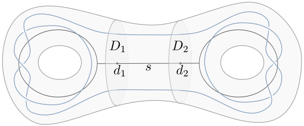

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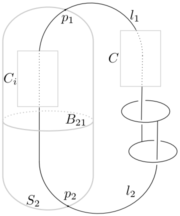
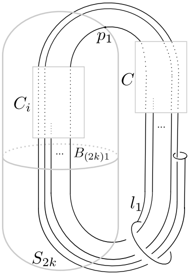
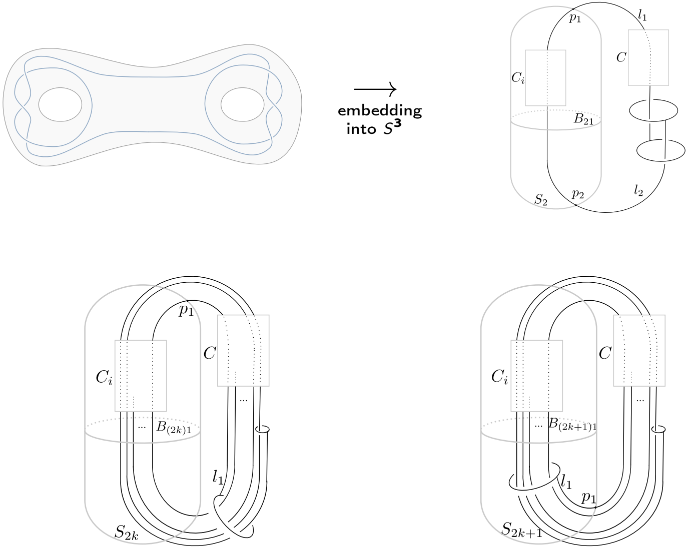




Knots with essential tangle decompositions with arbitrarily high number of strings



Remarks and questions

Infinite collection of prime knots such that each knot has a meridional planar essential surface in its complement for arbitrarily high number of boundary components.

The examples are satellite knots. There are hyperbolic knot examples with the same property obtained from similar ideas.

What characterizes knots with essential tangle decompositions with (without) arbitrarily high number of strings?

Are there knots with meridional essential surfaces of arbitrarily high genus on top of arbitrarily high number of boundary components? (Following sections.)

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Knot exterior with meridional essential surfaces of unbounded genus

Lyon, 71: Knot exterior with closed essential surfaces of arbitrarily high genus.

Question:

Is there a prime knot exterior with meridional essential surfaces of arbitrarily high genus and two boundary components?

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Knot exterior with meridional essential surfaces of unbounded genus

Theorem 2 (N., 15).
There is an infinite collection of prime knots such that

- for all $g \geq 1$
- each knot has a meridional essential surface of genus g and two boundary components.

Corollary.
There is an infinite collection of knots such that

- for all $g \geq 0$
- each knot has a meridional essential surface of genus g and two boundary components.

Knot exterior with meridional essential surfaces of unbounded genus

We should avoid:

- sattelite construction to obtain primeness, as we want the surfaces to have only two boundary components;
- use connected sum as the base for the construction, as we want the knots to be prime.

Strategy:
Construct the knot by identifying the boundaries of two solid tori containing, each, a properly embedded essential arc.

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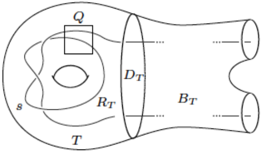

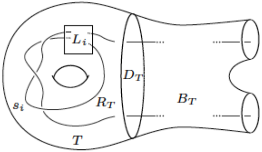

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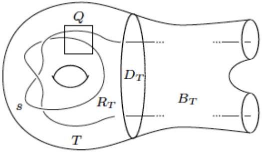

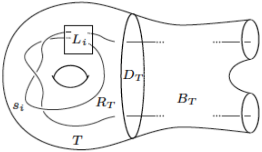

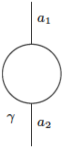

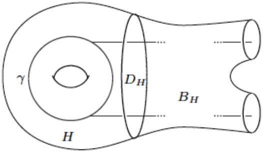

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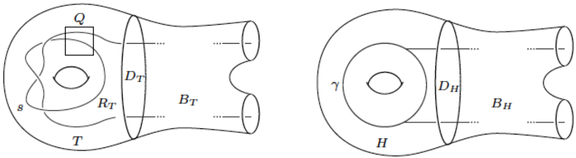

Lemma

- *The surfaces ∂H and ∂T are incompressible in $E_H(T)$;*
- *The arc s is essential in T;*
- *There is no properly embedded essential disk in $E_H(T)$ with boundary the union of an arc in ∂T and an arc ∂H, and not bounding a disk in $\partial E_H(T)$.*

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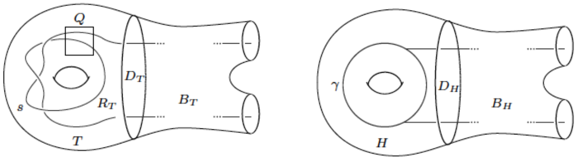

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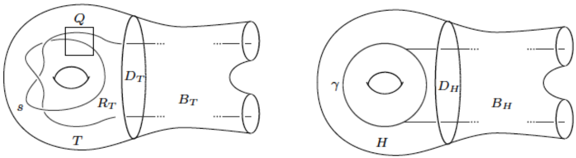

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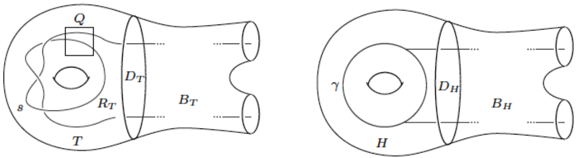

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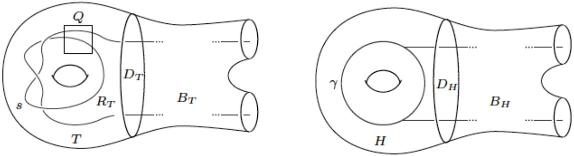

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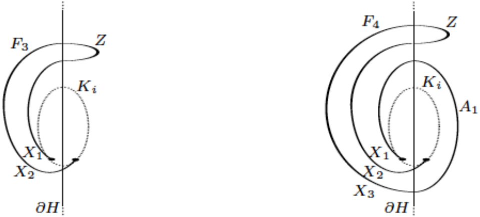

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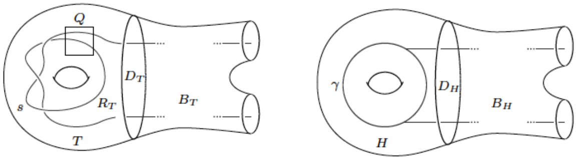

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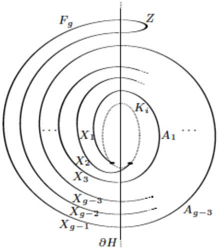
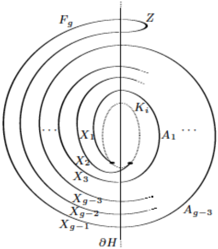


Knot exterior with meridional essential surfaces of unbounded genus



Branched surface:

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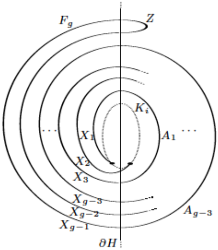

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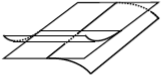

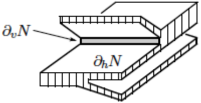

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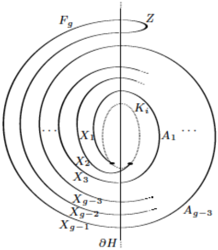

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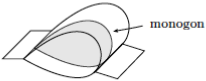

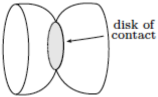

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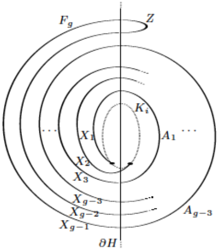

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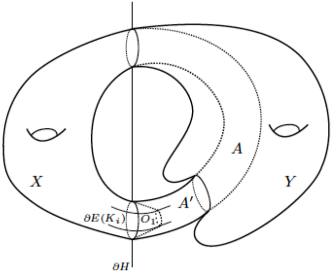

Knot exterior with meridional essential surfaces of unbounded genus and boundary components

We now know:

1. There is a knot exterior with meridional planar surfaces for arbitrarily high number of boundary components.
2. There is a knot exterior with meridional essential surfaces for arbitrarily high genus and two boundary components.

Question:

Can the genus and number of boundary components of meridional essential surfaces be simultaneously unbounded?

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Menasco, 84: In the complement of an alternating knot, for a fixed number of boundary components there are finitely many meridional essential surfaces.

Proposition. Let K be a knot which complement contains a meridional essential surface of genus g and n boundary components.
Then, there is a knot $h(K)$ which complement contains meridional essential surfaces of genus g and b boundary components for all even $b \geq 2n$.

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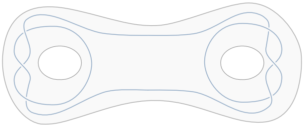
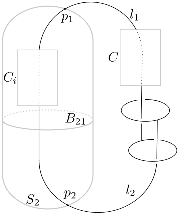


$\longrightarrow$



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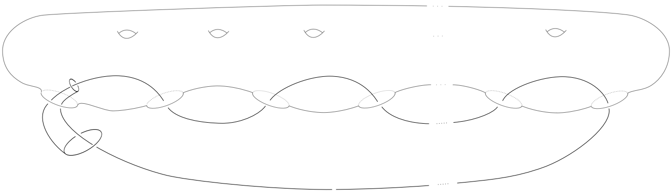
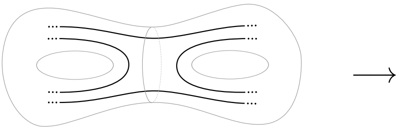
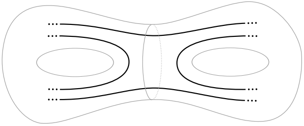


Sketch of proof



Connected sum of two knots with meridional essential surfaces with two boundary components and unbounded genus

Sketch of proof



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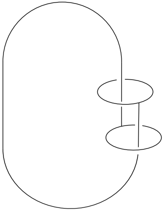

Handlebody-knot Γ

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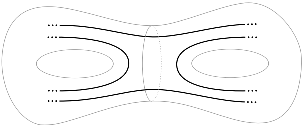

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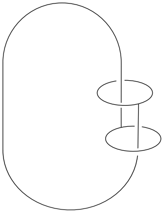

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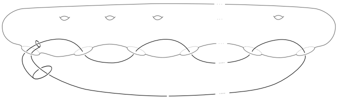

The knot $h(K)$ results from a sattelite of Γ with companion K.

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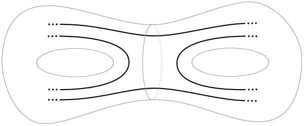

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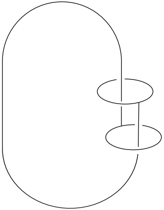

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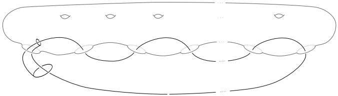

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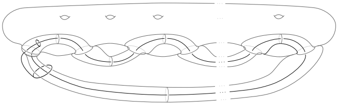

The branched surface carrying with positive weights the resulting meridional essential surfaces in the complement of $h(k)$.

Knot exterior with meridional essential surfaces of unbounded genus and boundary components

Theorem 3 (N., 15): There are infinitely many knots each of which having in its exterior meridional essential surfaces of all genus and $2n$ boundary components for all $n \geq 1$.

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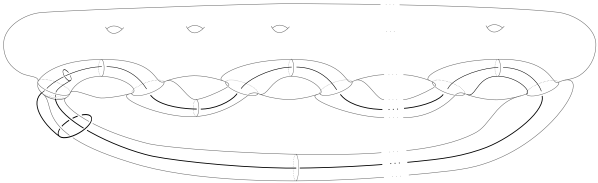

Choosing the knot K to have meridional essential surfaces of all genus and two boundary components, the sattelite $h(K)$ is as in the statement.

Knot exterior with meridional essential surfaces of unbounded genus and boundary components

Theorem 4 (N., 15): There are infinitely many hyperbolic knots each of which having in its exterior meridional essential surfaces of simultaneously unbounded genus and number of boundary components.

Knot exterior with meridional essential surfaces of unbounded genus and boundary components

Sketch of proof for hyperbolic knots: Consider a knot K as in the statement of Theorem 3.

From Myers, 82, let J be a null-homotopic knot in $E(K)$ with hyperbolic complement.

We proceed with $\frac{1}{r}$-Dehn filling on J and get a hyperbolic knot $K_r \subset S^3$.

From Boileau-Wang, 96, there is a degree-one map $f : E(K_r) \to E(K)$. This map preserves the meridional boundaries of the surfaces.

The restriction to surfaces $f : F_{g;b} \to S_{g;b}$ is also a degree-one map. From Edmonds, 79, it is a pinch map. The genus of $F_{g;b}$ is higher than of $S_{g;b}$.

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Meridional essential surfaces of unbounded Euler characteristics in knot complements

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University of Coimbra, Portugal

Institute for Advanced Study

April 20, 2016