Black hole degeneracies from worldsheet instantons:

From macroscopics to microscopics in string theory

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Two lessons have served me well

(Among all the others…)

1. “That had to be correct”

2. “That cannot be correct”

(I had prepared many interesting stories for this talk, but see Lesson 2.)

MICROSCOPIC KNOWLEDGE FROM MACROSCOPIC PHYSICS IN STRING THEORY

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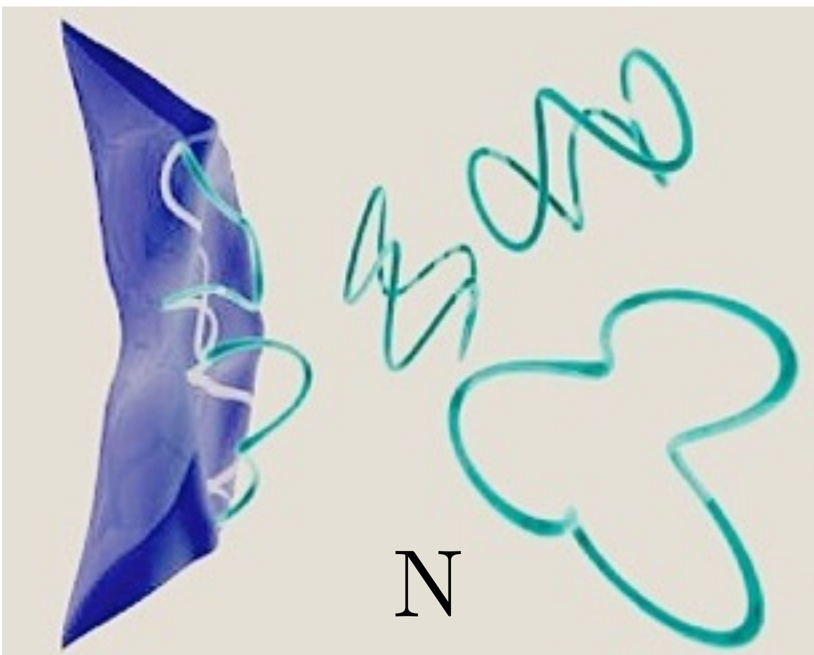
1. Introduction

Many interesting questions in string theory can be addressed by focussing on an effective lagrangian for the massless modes. This lagrangian is obtained by integrating out the infinite tower of massive states of the string. This is in the spirit of the wilsonian approach to field theory where the high energy modes can be integrated out and all the relevant information for low-energy physics is summarized in a low-energy effective action. This effective action is a powerful tool because it allows one to describe low-energy phenomena in a formulation where all the symmetries of the problem are manifest. These symmetries impose constraints on the form of the effective action which allow one to prove results about the microscopic theory. In particular, it is relatively easy in this framework to explore the consequences of spacetime supersymmetry. Such an approach has recently been used [1–5] to derive

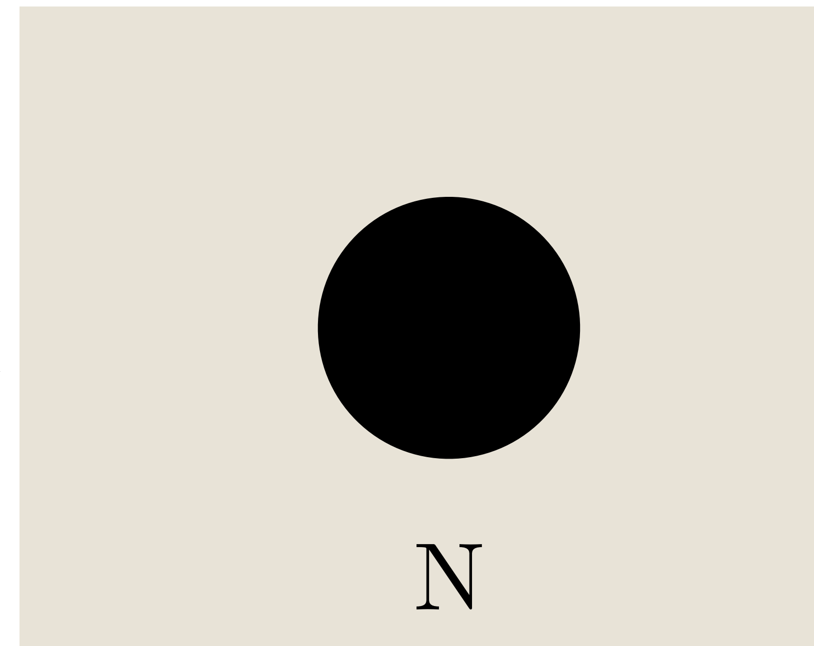
Black holes in string theory are ensembles of microscopic excitations of string theory

Microscopic



$g_s \mathrm{N} \ll 1$ ⟵ g_s ⟶ $g_s \mathrm{N} \gg 1$

Macroscopic



Strominger-Vafa '96

$$d_{\mathrm{micro}}(\mathrm{N}) = e^{\pi\sqrt{\mathrm{N}}} + \cdots \quad (\mathrm{N} \to \infty)$$

Bekenstein-Hawking '74

$$S_{\mathrm{BH}}^{\mathrm{class}} = \frac{A_{\mathrm{H}}}{4\,\ell_{\mathrm{Pl}}^2} = \pi\sqrt{\mathrm{N}}$$

Continuum physics appears via asymptotic expansion of integer degeneracy

$$\log d_{\mathrm{micro}} = S_{\mathrm{BH}}^{\mathrm{class}} + \cdots$$

⟶ $S_{\mathrm{BH}}^{\mathrm{quant}}$ (finite N)

Generating function of these microscopic degeneracies are modular forms

Prototype: 1/8 BPS BH in N=8 superstring theory in d=4.

U-duality $E_{7,7}(\mathbb{Z})$ invariant [M. Cvetic, D. Youm, '96]

$$N(\mathcal{Q}) = C^{abcd}\mathcal{Q}_a\mathcal{Q}_b\mathcal{Q}_c\mathcal{Q}_d \qquad a = 1, \cdots, 56$$

$$\sum_{n=-1\,(\mathrm{mod}\,4)} d_{\mathrm{micro}}(n)\, q^{n/4} = \theta_0(q)/\eta(q)^6 = q^{-1/4}(1+8q+39q^2+152q^3+\cdots)$$

$$\sum_{n=0\,(\mathrm{mod}\,4)} d_{\mathrm{micro}}(n)\, q^{n/4} = \theta_1(q)/\eta(q)^6 = 2+12q+56q^2+208q^3+\cdots$$

$$\theta_a(q) = \sum_{n\in\mathbb{Z}+a/2} q^{n^2} \qquad \eta(q) = q^{1/24}\prod_{n\geq 1}(1-q)^n$$

Such functions are extremely specia
and they have been studied by
mathematicians for more than a cen

[J. Maldacena, G. Moore, A. Strominger '99]

Modular symmetry → Microscopic degeneracy is a sum of Bessel functions

[Analytic number theory: G. Hardy, S. Ramanujan; H. Rademacher]

$$
\begin{aligned}
d_{\mathrm{micro}}(N) &= \sum_{c=1}^{\infty} c^{-9/2} K_c(N)\, \widetilde{I}_{7/2}\big(\frac{\pi\sqrt{N}}{c}\big) \\
&= \widetilde{I}_{7/2}(\pi\sqrt{N})\Big(1 + O(e^{-\pi\sqrt{N}/2})\Big) \\
&= e^{\pi\sqrt{N}}\left(1 - \frac{15}{4}\log N + O(\frac{1}{N})\right).
\end{aligned}
$$

$K_c(N)$ Kloosterman sum

$\widetilde{I}_\rho(z) = 2\pi\big(\frac{z}{4\pi}\big)^{-\rho} I_\rho(z)$ I-Bessel function

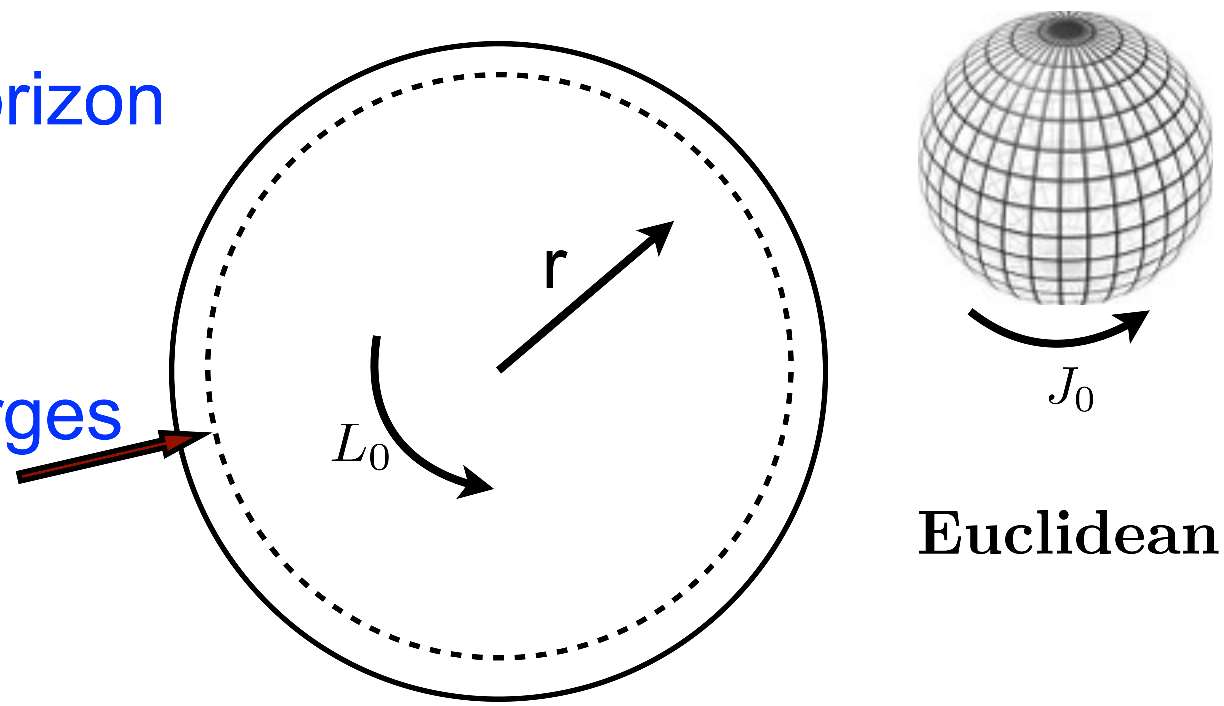
Note: only input is one polar term!

Quantum BPS black hole entropy is an $\mathrm{AdS_2}$ functional integral

(Sen '08, '09)

4d BPS BH Near-horizon

B.C.s fixed by charges
(classical attractor)



Supersymmetry

$Q^2 = L_0 - J_0 \, .$

Euclidean $\mathbf{AdS_2 \times S^2}$

$$\exp(S^{\rm qu}_{BH}(q_I)) \equiv Z_{AdS_2}(q_I) = \left\langle \exp\left[-i\, q_I \oint A^I \right] \right\rangle^{\rm reg}_{\rm AdS_2}$$

Saddle point evaluation $\Longrightarrow$ Classical Wald entropy.

Integer BH degeneracy can be computed from Macroscopic (continuum) physics

(A.Dabholkar, J.Gomes, S.M. '10, '11, '14)

$$
\begin{aligned}
\exp(S_{\rm BH}^{\rm quant}(N)) &= \sum_{c=1}^{\infty} c^{-9/2} K_c(N)\, \widetilde{I}_{7/2}(\frac{\pi\sqrt{N}}{c}) \\
&= \widetilde{I}_{7/2}(\pi\sqrt{N})\big(1 + O(e^{-\pi\sqrt{N}/2})\big) \\
&= e^{\pi\sqrt{N}}\Big(1 - \frac{15}{4}\log N + O(\frac{1}{N})\Big) \,.
\end{aligned}
$$

Orbifolds of AdS_2

All-order perturbation using localization

Bekenstein-Hawking

(Maldacena, Moore, Strominger '99)

One-loop corrections

(Sen, Banerjee, Gupta, Mandal '11)

Comparison in N=8 string theory

N	$d_{\text{micro}}(\text{N})$		$\exp\left(S^{\text{cl}}(\text{N})\right)$
3	8		230.76
4	12		535.49
7	39		4071.93
8	56		7228.35
11	152		33506.14
12	208		53252.29
15	513		192400.81
...	...		...
10^5	exp(295.7)		exp(314.2)

$$\log\left(d_{\text{micro}}\right) \xrightarrow{N\to\infty} S^{\text{cl}}_{BH} \,.$$

Comparison in N=8 string theory

(A.Dabholkar, J.Gomes, S.M. '11)

N	$d_{\rm micro}(\rm N)$	$\exp\big(S^{\rm qu}(\rm N)\big)$	$\exp\big(S^{\rm cl}(\rm N)\big)$
3	8	7.97	230.76
4	12	12.2	535.49
7	39	38.99	4071.93
8	56	55.72	7228.35
11	152	152.04	33506.14
12	208	208.45	53252.29
15	513	512.96	192400.81
...	...	...	...
10^5	exp(295.7)	exp(295.7)	exp(314.2)

$$d_{\rm micro}(N) = e^{S^{\rm qu}_{BH}(N)}\big(1 + O(e^{-\pi\sqrt{N}/2})\big)$$

Gravitational corrections are governed by Macroscopic effective action

- Quantum effects in gravity.
- Input: Wilsonian effective action of light modes (graviton+…) (e.g. using supersymmetry/string theory).
- Compute functional integral over light modes.

Supersymmetric Localization

Duistermaat-Heckmann, Atiyah-Singer-Bott, Berligne-Vergne, Witten (1980s), Pestun ’07

For prototype N=8 theory, effective action is very simple (tree level F-terms are exact).

How generic are these ideas (Macro)?

In theories with lower supersymmetry:

1. Stringy effects (instantons) contribute to (F-term) effective action.

cf. Introduction

Q1. How do the instanton degeneracies encode the BH degeneracies?

Microscopic degeneracies in N=4 string theory (Type II on $K3 \times T^2$) are known

Partition function is the inverse of the Igusa cusp form

$$\frac{1}{\Phi_{10}(\tau, z, \sigma)} = \sum_{n,\ell,m} d(n, \ell, m)\, e^{2\pi i(n\tau + \ell z + m\sigma)}$$

1/4 BPS states

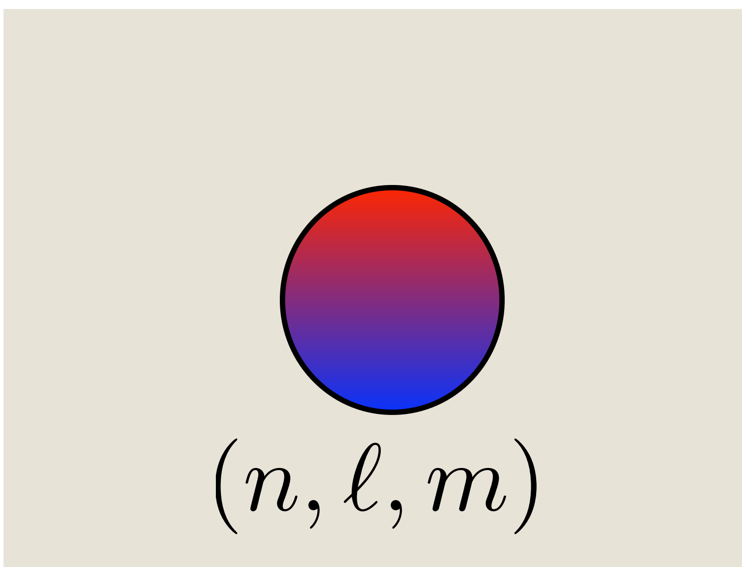
(R. Dijkgraaf, E.+H. Verlinde, 1994)
(D. Shih, A. Strominger, X. Yin '05; J. David, A. Sen '06)

Fourier Expansion is ill-defined due to meromorphicity!

$$\Phi_{10}(\tau, z, \sigma) = 4\pi z^2 \eta(\tau)^{24}\, \eta(\sigma)^{24} + O(z^4)$$

“Phenomenology” of the N=4 theory (Meaning of ambiguity in physics)

$\frac{1}{4}$-BPS dyonic BH

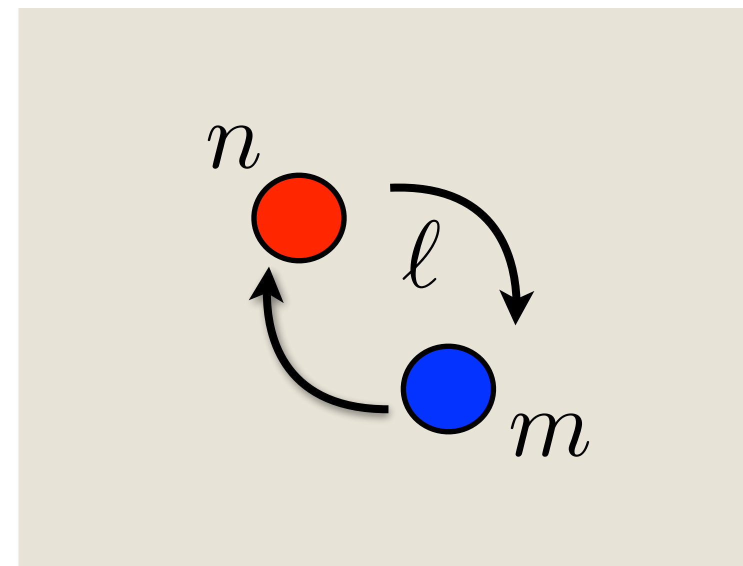


$$d_{\text{BH}}(m, \ell, n) \approx e^{\pi\sqrt{4mn-\ell^2}}$$

Exists everywhere in moduli space

2-centered BH bound state

(Each $\frac{1}{2}$-BPS)



$$d^{(2)}(m, \ell, n) = p_{24}(m+1)\, p_{24}(n+1)\, \ell$$
$$\approx e^{4\pi(\sqrt{n}+\sqrt{m})}$$

(Dis)appears on crossing a co-dimension one surface (wall) in moduli space

(Denef ’00; Cheng, Verlinde ’07)

One can isolate the BH degeneracies

$$\frac{1}{\Phi_{10}(\tau,z,\sigma)} = \sum_{m=-1}^{\infty} \psi_m(\tau,z)\, e^{2\pi i m\sigma}$$

Expansion in M-theory limit

$\psi_m(\tau,z)$ Jacobi form of index m *meromorphic in z!*

Canonical decomposition (A.Dabholkar, S.M., D. Zagier '12)

$$\psi_m(\tau,z) = \psi_m^{\mathrm{BH}}(\tau,z) + \psi_m^{\mathrm{multi}}(\tau,z)\,.$$

Partition function of the isolated BH is a *mock Jacobi form*.

Multi-centered BH contribution.

Practical implication of *mock* nature

Mock means that the function itself is not quite modular, but one can add a specific non-holomorphic function (called the shadow function) to it so that the sum is modular (but not holomorphic).

So the power of modularity is resurrected!

(S. Ramanujan 1920, S. Zwegers ’02)

In particular, there is a Rademacher-type formula for mock Jacobi forms, but with some modifications from the modular case.

(K. Bringmann, K.Ono ’07)

Microscopic degeneracy formula (from mock modular form)

$$c_m(n,\ell) \approx \sum_{0 \le \ell' \le m} \sum_{4n' - \frac{\ell'^2}{m} < 0} c_m(n',\ell') \cos\left(\pi(m-\ell')\frac{\ell}{m}\right) \times \frac{2\pi}{\sqrt{m}} \left(\frac{\left|4n' - \frac{\ell'^2}{m}\right|}{n - \frac{\ell^2}{4m}} \right)^{23/4} I_{23/2}\left(2\pi \sqrt{\left|4n' - \frac{\ell'^2}{m}\right| \left(n - \frac{\ell^2}{4m}\right)} \right)$$

Polar coefficients of mock Jacobi form

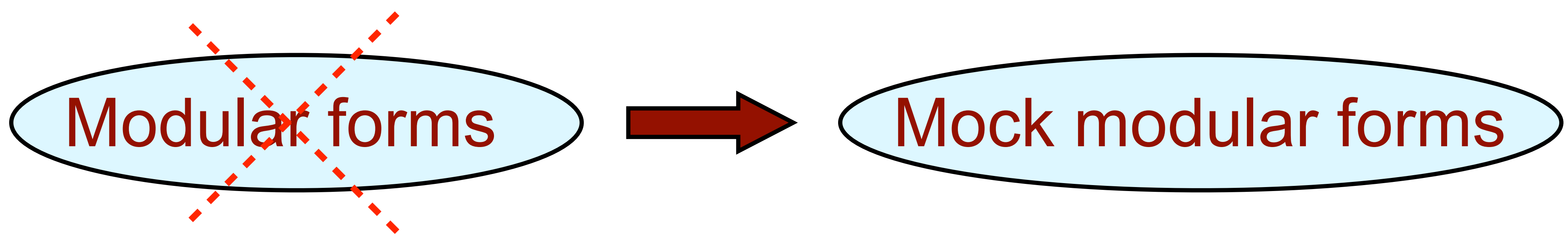
(This is the c=1 term of the Rademacher expansion for true Jacobi forms, one can calculate the corrections due to the mock nature.)

How generic are these ideas (Macro/micro)?

In theories with lower supersymmetry:

1. Stringy effects (instantons) in gravitational effective action.

2. Black hole generically breaks Modular symmetry.



N=4 string theory

Q1. How do the instanton degeneracies encode the BH degeneracies?

Q2. How does gravity see the mock modular symmetry?

Where we are headed: the beginning of a formula (still in progress)

In the context of compactification of Type II string theory on $M_6 = K3 \times T^2$, this formula gives a simple relation between the degeneracies of worldsheet instantons on M_6 — the Gromov-Witten invariants — and the degeneracies of single-centered BHs.

cf. OSV, Denef-Moore

Based on: S.M., V. Reys arXiv:1512.01553

and on the work of:

Cardoso, de Wit, Kaeppelli, Mohaupt (first computations of corrections to BH formula in string theory),
Sen (Quantum entropy function program),
Ooguri, Strominger, Vafa (OSV formula)

Banerjee, Dabholkar, David, Denef, Gaiotto, Gomes, Gupta, Hama, Hosomichi, Jatkar, Lal, Mahapatra, Mandal, Moore, Pestun, Pioline, Shih, Yin, …

Beyond large N: exact functional integral

Localization

$$I := \int_{\mathcal{M}} d\mu\, \mathcal{O}\, e^{-\mathcal{S}} \quad \Rightarrow \quad I = \int_{\mathcal{M}_Q} d\mu_Q\, \mathcal{O}\, e^{-\mathcal{S}}\, Z_{\text{1-loop}}(Q\mathcal{V})$$

$$\mathcal{M}_Q = \{Q\Psi = 0\}$$

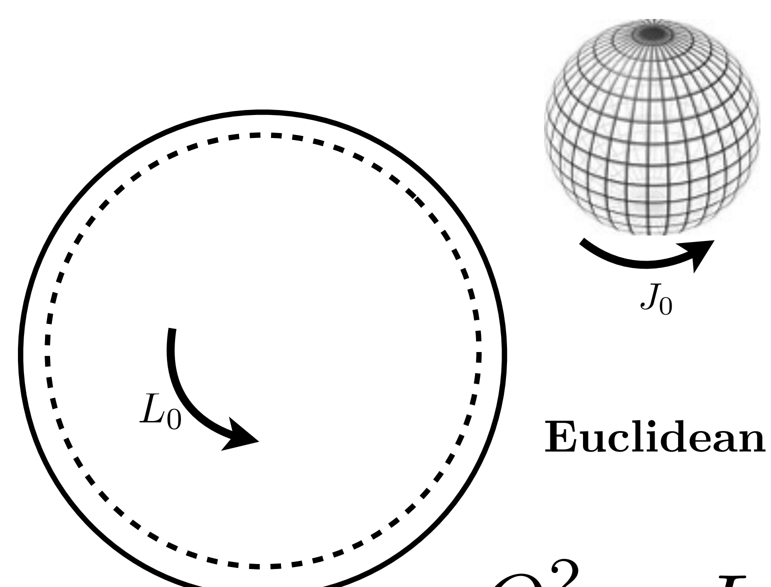
Deformation term

Huge reduction!

$$Z_{\text{AdS}_2}(q_I) \equiv \int_{\mathcal{M}} d\mu\, \mathcal{O}\, e^{-\mathcal{S}}$$

Supergravity field space

Effective action



Euclidean $\mathbf{AdS_2 \times S^2}$

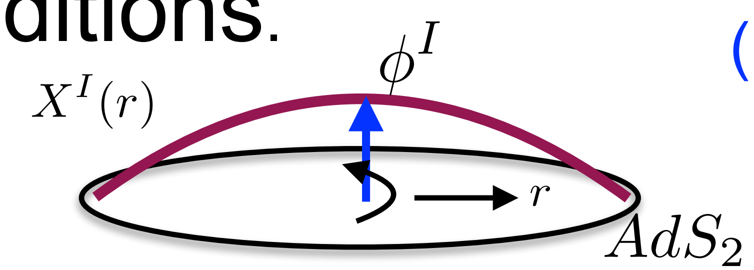
$$Q^2 = L_0 - J_0\,.$$

Localization in supergravity

(A.Dabholkar, J.Gomes, S.M. '10, '11, '14)

1. Formalism with off-shell susy: d=4, N=2 graviton + $n_{\rm v}$ vector multiplets. (de Wit, van Holten, Van Proeyen '80)

2. All solutions of localization equations $Q\,\Psi = 0$, w/ $AdS_2 \times S^2$ boundary conditions. (DGM '10, R.Gupta, S.M. '12)



3. Evaluate full supergravity action on these solutions (include all higher derivative terms). D-terms do not contribute! (S.M., V.Reys, '13)

4. Compute determinant and measure.

(Cardoso, de Wit, Mahapatra '12, S.M., V. Reys, '15, Y. Ito, R. Gupta, I. Jeon,'15)

+ ongoing work w/ I. Jeon, V. Reys, B. de Wit. cf Dedushenko, Witten '14.

Simple formula for exact entropy of $\frac{1}{2}$-BPS BH in theories with 8 supercharges

(A.Dabholkar, J.Gomes, S.M. ’10, ’11, ’14)

4d N=2 supergravity coupled to $n_{\rm v}$ vector multiplets,
BH carrying charges (p^I, q_I) $I = 0, 1, \cdots, n_{\rm v}$

$$Z_{AdS_2}(q,p) = \int \prod_{I=0}^{n_{\rm v}} [d\phi^I] \exp(\mathcal{S}_{\rm ren}(\phi, p, q))$$

$$\mathcal{S}_{\rm ren}(\phi, p, q) = -\pi q_I \phi^I + {\rm Im} F(\phi^I + i p^I)$$

Here the function $F(X^I)$ is the holomorphic prepotential of N=2 supergravity.

(c.f. Ooguri-Strominger-Vafa ’04)

Prototype: 1/8 BPS black holes in N=8 string theory

- Truncation of N=8 to N=2 theory with 7 vectors.
- F-term action (prepotential) exact at tree-level.

$$F(X) = -\frac{1}{2}\,\frac{X^1 C_{ab} X^a X^b}{X^0}\,, \qquad a,b = 2,\ldots,7\,.$$

⬇

Exact quantum gravitational entropy

$$e^{S_{BH}^{\rm qu}}(N) = \int \frac{d\sigma}{\sigma^{9/2}} \exp\left(\sigma + \pi^2 N/4\sigma\right) = \widetilde{I}_{7/2}(\pi\sqrt{N})$$

Type II string theory/ $K3 \times T^2$: Macroscopics

- F-term action (prepotential) receives contributions from worldsheet instantons.

$$F(X) = -\frac{X^1 X^a C_{ab} X^b}{X^0} + \frac{1}{2\pi i}\mathcal{F}^{(1)}(X^1/X^0)$$

$$(a, b = 2, \cdots, 23)$$

$$\mathcal{F}^{(1)}(\tau) = \log \eta(\tau)^{24}$$

Instanton contributions

Fourier expansion gives the instanton degeneracies $(q = e^{2\pi i \tau})$

$$e^{-\mathcal{F}^{(1)}(\tau)} = \sum_{n=-1}^{\infty} d(n)\, q^n \quad = q^{-1} + 24 + 324q + 3200q^2 + \cdots$$

Macroscopic quantum BH entropy

- Using this expansion in our supergravity formula

$$Z_{AdS_2}(q,p) = \int \prod_{I=0}^{n_{\mathrm{v}}} [d\phi^I] \exp(-\pi q_I \phi^I + \mathrm{Im} F(\phi^I + ip^I))$$

we get a series of Bessel functions.

Assume a certain measure factor (Full first-principles derivation of measure remains to be done).

- Apparently divergent. This problem has been analyzed in J.Gomes arXiv:1511.07061. A good integration contour was found that ensures convergence of this series.

Quantum BH entropy formula in N=4 theory

S.M., V.Reys arXiv:1512.01553

We obtain a sum over Bessel functions with numerical coefficients depending on the instanton degeneracies

$$Z_{AdS_2}(n,\ell,m) \approx \sum_{0\le \ell' \le m} \sum_{4n' - \frac{\ell'^2}{m} < 0} (\ell' - 2n')\, d(m+n'-\ell')\, d(n') \cos\left(\pi(m-\ell')\frac{\ell}{m}\right) \times \\ \times \frac{2\pi}{\sqrt{m}} \left(\frac{\left|4n' - \frac{\ell'^2}{m}\right|}{n - \frac{\ell^2}{4m}} \right)^{23/4} I_{23/2}\left(2\pi \sqrt{\left|4n' - \frac{\ell'^2}{m}\right| \left(n - \frac{\ell^2}{4m}\right)} \right)$$

1/4 BPS BH in N=4 theory

(This formula receives corrections from subleading saddle points, and from certain “edge terms” that are not written here.)

Microscopic vs macroscopic formula

In each case a sum over Bessel functions with some numerical coefficients (the polar coefficients)

$$c_m(n,\ell) \approx \sum_{0\le \ell' \le m} \sum_{4n'-\frac{\ell'^2}{m}<0} c_m(n',\ell') \cos\left(\pi(m-\ell')\frac{\ell}{m}\right) \times$$
$$\times \frac{2\pi}{\sqrt{m}} \left(\frac{\left|4n'-\frac{\ell'^2}{m}\right|}{n-\frac{\ell^2}{4m}}\right)^{23/4} I_{23/2}\left(2\pi\sqrt{\left|4n'-\frac{\ell'^2}{m}\right|\left(n-\frac{\ell^2}{4m}\right)}\right)$$

$$Z_{AdS_2}(n,\ell,m) \approx \sum_{0\le \ell' \le m} \sum_{4n'-\frac{\ell'^2}{m}<0} (\ell'-2n')\, d(m+n'-\ell')\, d(n') \cos\left(\pi(m-\ell')\frac{\ell}{m}\right) \times$$
$$\times \frac{2\pi}{\sqrt{m}} \left(\frac{\left|4n'-\frac{\ell'^2}{m}\right|}{n-\frac{\ell^2}{4m}}\right)^{23/4} I_{23/2}\left(2\pi\sqrt{\left|4n'-\frac{\ell'^2}{m}\right|\left(n-\frac{\ell^2}{4m}\right)}\right)$$

Macroscopic Prediction relating instanton degeneracies and BH degeneracies

S.M., V.Reys arXiv:1512.01553

Single-centered BH (polar degeneracies)

Instanton degeneracies

$$c_m(n,\ell) = (\ell - 2n)\, d(m+n-\ell)\, d(n)$$

$$4mn - \ell^2 < 0, \quad n \geq -1,\ 0 \leq \ell \leq m$$

This formula can still get corrections from lower order terms on both sides, that we have not calculated yet.

The mock modular forms encoding the N=4 BH degeneracies are explicitly known

(A.Dabholkar, S.M. D. Zagier '12) (K. Bringmann, S.M.'12)

m=1

$$\psi_1^{\mathrm{F}}(\tau,z) = \frac{1}{\eta(\tau)^{24}}\left(3E_4(\tau)A(\tau,z) - 648\mathcal{H}_1(\tau,z)\right)$$

$$= (3\zeta + 48 + 3\zeta^{-1})q^{-1} + (48\zeta^2 + 600\zeta - 648 + 600\zeta^{-1} + 48\zeta^{-2}) + \cdots$$

m=2

$$\psi_2^{\mathrm{F}}(\tau,z) = \frac{1}{3\eta(\tau)^{24}}\left(22E_4AB - 10E_6A^2 - 9600\mathcal{H}_2\right)$$

and so on …

Checks of prediction

$\mathbf{m = 1}$:

$\Delta = 4mn - \ell^2$

Δ	(n,ℓ)	$c_1(n,\ell)$	$(\ell - 2n)\, d(1+n-\ell)\, d(n)$
-5	$(-1,1)$	3	3
-4	$(-1,0)$	$\mathbf{48}$	48
-1	$(0,1)$	600	576

$\mathbf{m = 2}$:

Δ	(n,ℓ)	$c_2(n,\ell)$	$(\ell - 2n)\, d(2+n-\ell)\, d(n)$
-12	$(-1,2)$	4	4
-9	$(-1,1)$	72	72
-8	$(-1,0)$	$\mathbf{648}$	648
-4	$(0,2)$	1152	1152
-1	$(0,1)$	8376	7776

Checks of prediction

$\mathbf{m = 3}$:

Δ	(n,ℓ)	$c_3(n,\ell)$	$(\ell - 2n)\, d(3+n-\ell)\, d(n)$
-21	$(-1,3)$	5	5
-16	$(-1,2)$	96	96
-13	$(-1,1)$	972	972
-12	$(-1,0)$	$\mathbf{6404}$	6400
-9	$(0,3)$	1728	1728
-4	$(0,2)$	15600	15552
-1	$(0,1)$	85176	76800

We checked this up to m=7 (in principle we can continue).

In each case, the formula agrees in its regime of validity

Checks of prediction

$\mathbf{m = 7}$:

Δ	(n, ℓ)	$c_7(n, \ell)$	$(\ell - 2n)\, d(7 + n - \ell)\, d(n)$
-77	$(-1, 7)$	9	9
-64	$(-1, 6)$	192	192
-53	$(-1, 5)$	2268	2268
-49	$(0, 7)$	4032	4032
-44	$(-1, 4)$	19200	19200
-37	$(-1, 3)$	128250	128250
-36	$(0, 6)$	46656	46656
-32	$(-1, 2)$	705030	705024
-29	$(-1, 1)$	3222780	3221160
-28	$(-1, 0)$	**11963592**	11860992
-25	$(0, 5)$	384000	384000
-21	$(1, 7)$	524880	524880
-16	$(0, 4)$	2462496	2462400
-9	$(0, 3)$	12713760	12690432
-8	$(1, 6)$	4147848	4147200
-4	$(0, 2)$	52785360	51538560
-1	$(0, 1)$	173032104	142331904

Modification due to mock nature

Expect corrections to sugra formula

Lessons and outlook

- Degrees of freedom of a BH are encoded in an intricate manner in gravity. Modular symmetry is useful to decode.
- Gravity path integral seems to know about mock nature.
- Instantons in supergravity encode the BH degeneracies via an explicit and simple relation. Need to derive a complete formula!
- Constraints on low energy Lagrangian.

Thank you for your attention!

Happy birthday Nati!

Some more details

Localization in supergravity: the steps.

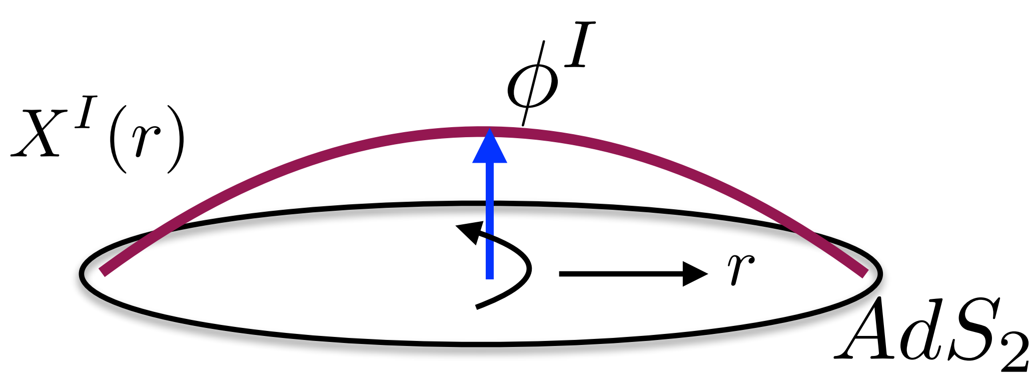
(A.Dabholkar, J.Gomes, S.M. '10, '11, '14)

0. Formalism: 4d N=2 off-shell conformal supergravity coupled to vector multiplets.

(de Wit, van Holten, Van Proeyen '80)

1. All solutions of localization equations $Q\,\Psi = 0$, subject to $AdS_2 \times S^2$ boundary conditions.

- In vector multiplet sector, scalar fields go off-shell:



(A.Dabholkar, J.Gomes, S.M. '10)

- In the gravity multiplet, conformal mode of graviton goes off-shell.

(R.Gupta, S.M. '12)

The steps (contd.)

$$I = \int_{\mathcal{M}_Q} d\mu_Q \, \mathcal{O} \, e^{-\mathcal{S}} \, Z_{\text{1-loop}}(Q\mathcal{V})$$

$$= \{Q\Psi = 0\}$$

3. Generic supergravity action contains chiral-superspace integrals (F) and full-superspace integrals (D).

   Only holomorphic chiral-superspace integrals contribute. (These terms are governed by holomorphic prepotential $F(X^I)$ computable in string theory.) (S.M., V.Reys, ’13)

4. Compute determinant and measure.

(Cardoso, de Wit, Mahapatra ’12, S.M., V. Reys, ’15, Y. Ito, R. Gupta, I. Jeon, ’15)

+ ongoing work w/ I. Jeon, V. Reys, B. de Wit. cf Dedushenko, Witten ’14.

Jacobi forms

Jacobi form of weight k, index m

$\varphi(\tau, z)$ Holomorphic function $\mathbb{H} \times \mathbb{C} \to \mathbb{C}$

Modular property:

$$\varphi\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) \;=\; (c\tau + d)^k \, e^{\frac{2\pi i m c z^2}{c\tau + d}} \, \varphi(\tau, z)$$

Elliptic property:

$$\varphi(\tau, z + \lambda\tau + \mu) \;=\; e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\varphi(\tau, z) \quad \forall\, \lambda,\, \mu \in \mathbb{Z}$$

Jacobi forms have a Fourier expansion

Weak Jacobi forms

$$\varphi(\tau, z) \;=\; \sum_{n\geq 0,\ell} c(n,\ell)\, q^n\, \zeta^\ell$$

$$q = e^{2\pi i\tau}$$
$$\zeta = e^{2\pi i z}$$

Elliptic property ➡

$$c(n,\ell) \;=\; C_\mu(\Delta)$$

$$\Delta = 4nm - \ell^2$$
$$\mu = \ell \pmod{2m}$$

Special coefficients: the *polar coefficients*

$C_\mu(\Delta)$ with $\Delta < 0$

Polar coefficients completely determine the Jacobi form

$$\varphi(\tau, z) = \sum_{n \geq 0, \ell} c(n, \ell)\, q^n\, \zeta^\ell$$

weight $k = w + 1/2$

index m

Hardy-Ramanujan-Rademacher expansion

$$C_\ell(\Delta) = (2\pi)^{2-w} \sum_{c=1}^{\infty} c^{w-2} \sum_{\widetilde{\ell}\,(\mathrm{mod} 2m)} \sum_{\widetilde{\Delta}<0} C_{\widetilde{\ell}}(\widetilde{\Delta})\, Kl(\Delta, \ell, \widetilde{\Delta}, \widetilde{\ell}; c) \times$$

$$\times \left| \frac{\widetilde{\Delta}}{4m} \right|^{1-w} \widetilde{I}_{1-w}\Big(\frac{\pi}{mc} \sqrt{|\widetilde{\Delta}|\Delta} \Big)$$

Only input: Polar coefficients

Mock modular forms

S. Ramanujan (1920) — S. Zwegers (2002)

Mock modular form $f(\tau)$ ⟺ *Shadow* $g(\tau) \in M_{2-k}$

Completion $\widehat{f}(\tau,\bar\tau) := f(\tau) + g^*(\tau,\bar\tau)$

transforms like a modular form of weight k,

where $g^*(\tau) = \left(\frac{i}{2\pi}\right)^{k-1} \int_{-\bar\tau}^{\infty} (z+\tau)^{-k}\, \overline{g(-\bar z)}\, dz$

$$g(\tau) = \sum_{n>0} b_n\, q^n \Rightarrow \quad g^*(\tau) = \sum_{n>0} n^{k-1}\, \bar b_n\, \Gamma(1-k, 4\pi n \tau_2)\, q^{-n}$$

⇒ *Holomorphic anomaly equation*

$$(4\pi\tau_2)^k \frac{\partial \widehat{f}(\tau,\bar\tau)}{\partial\bar\tau} = -2\pi i\, \overline{g(\tau)}\,.$$

Checks of prediction

The mock modular forms encoding the N=4 BH degeneracies are explicitly known (A.Dabholkar, S.M. D. Zagier '12) (K. Bringmann, S.M.'12)

$\mathbf{m = 1}$:

$\Delta = 4mn - \ell^2$

Δ	(n, ℓ)	$c_1(n, \ell)$	$(\ell - 2n)\, d(1 + n - \ell)\, d(n)$
-5	$(-1, 1)$	3	3
-4	$(-1, 0)$	**48**	48
-1	$(0, 1)$	600	576

$\mathbf{m = 2}$:

Δ	(n, ℓ)	$c_2(n, \ell)$	$(\ell - 2n)\, d(2 + n - \ell)\, d(n)$
-12	$(-1, 2)$	4	4
-9	$(-1, 1)$	72	72
-8	$(-1, 0)$	**648**	648
-4	$(0, 2)$	1152	1152
-1	$(0, 1)$	8376	7776