Arithmetic regularity, removal, and progressions

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Marston Morse Lecture Series

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Roth's theorem

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Behrend construction gives a lower bound of $\frac{N}{e^{c\sqrt{\log N}}}$.

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Bateman-Katz: $|A| = O(3^n/n^{1+c})$.

Breakthrough

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Same conclusion for the *multicolored sum-free problem*: If $\{x_i\}_{i=1}^m, \{y_i\}_{i=1}^m, \{z_i\}_{i=1}^m \subset \mathbb{F}_p^n$ with $x_i + y_j + z_k = 0 \Leftrightarrow i = j = k$, then $m \leq p^{(1-c_p)n}$.

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Theorem

Exponent is sharp for the *multicolored sum-free problem*: for $\mathbb{F}_2$ by construction of Fu-Kleinberg, $\mathbb{F}_p$ by Kleinberg-Sawin-Speyer.

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Claim

Diagonal tensor has rank equal to number of nonzero elements.

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$$m \leq 3|M_n^{(p-1)n/3}|$$

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Take a tensor $T : (\mathbb{F}_p^n)^3 \to \mathbb{F}_p$:

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T is diagonal on $X \times Y \times Z$, so slice rank is at least m, and is at most $3|M_n^{(p-1)n/3}|$.

Arithmetic Regularity Lemma

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for every codimension 1 affine subspace T of $S + x$.

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For each $\varepsilon > 0$ there is $M(\varepsilon)$ such that for any $A \subset \mathbb{F}_3^n$, there is an ε-regular subspace S of codimension at most $M(\varepsilon)$.

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Green, Hosseini-Lovett-Moshkovitz-Shapira:
$M(\varepsilon)$ is a tower of twos of height $\varepsilon^{-O(1)}$.

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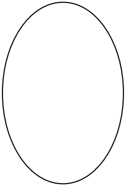
Problem (Green)

Improve the bound in the arithmetic triangle removal lemma.

Král', Serra, Vena proof

$\mathbb{F}_p^n$



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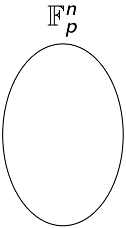

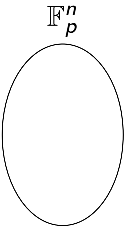

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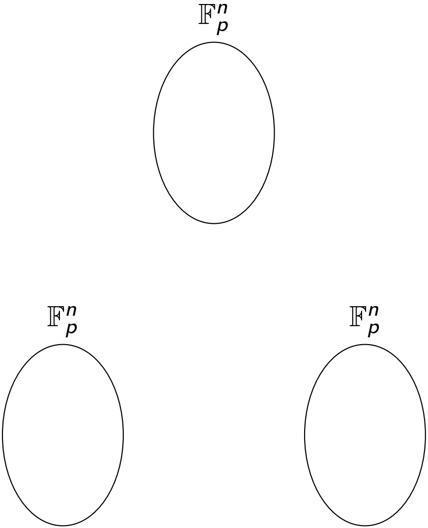

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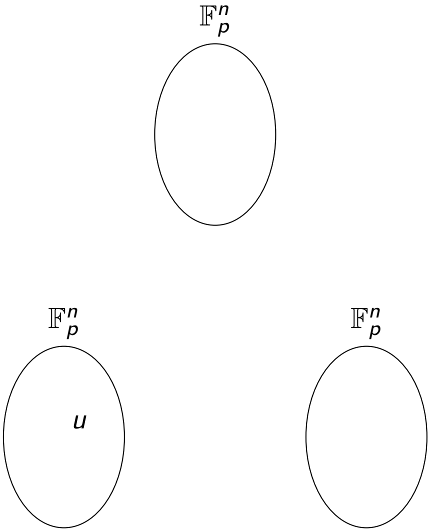

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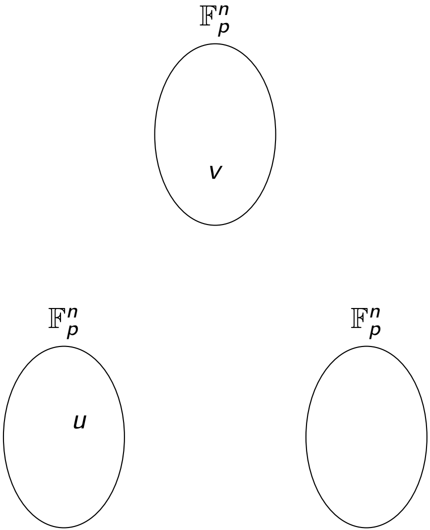

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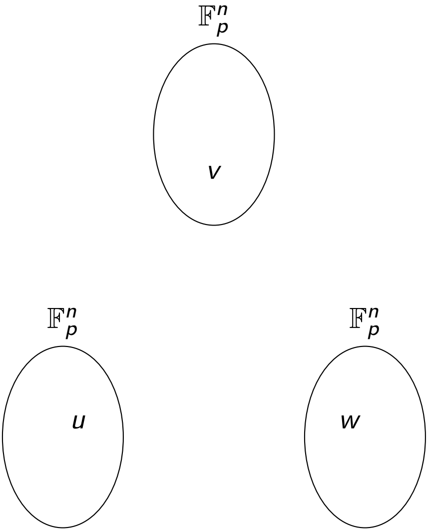

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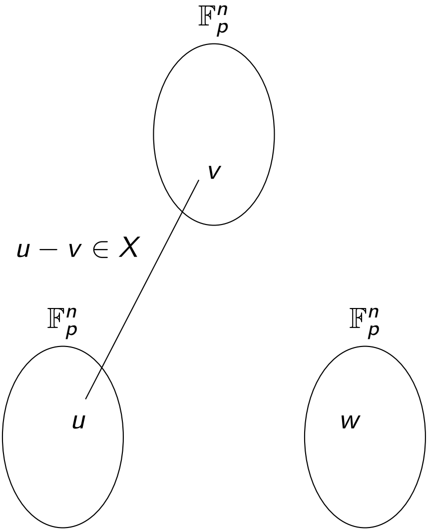

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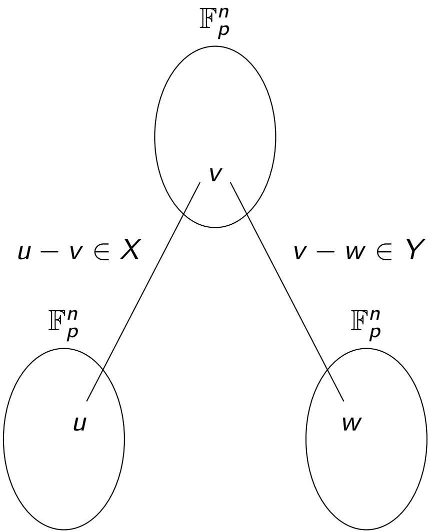

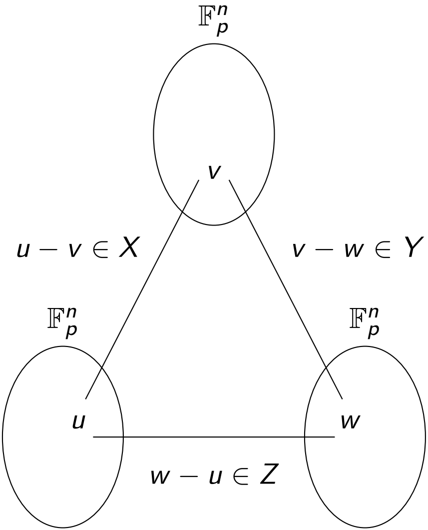
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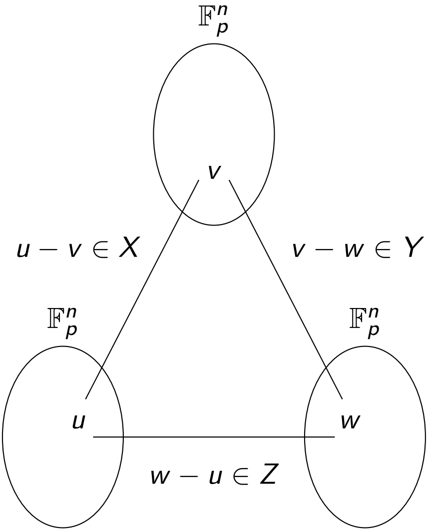

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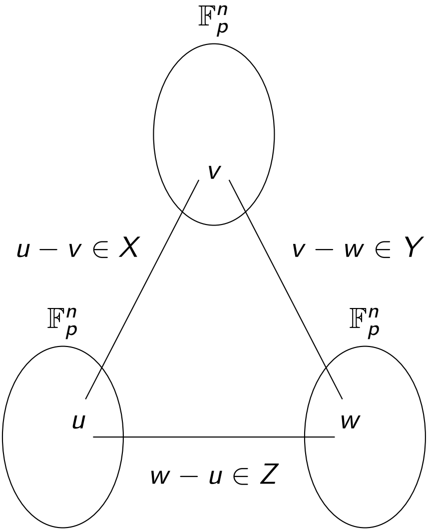

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Thus, there are at most δN^3 triangles.

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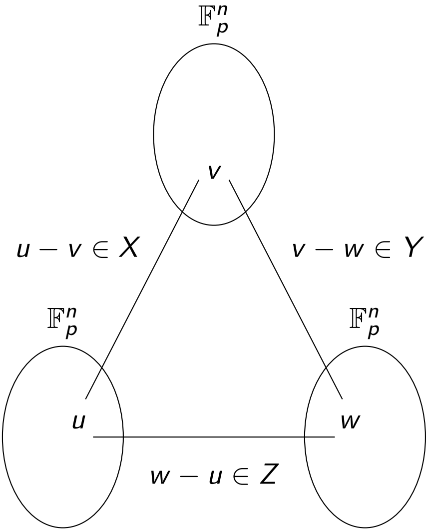

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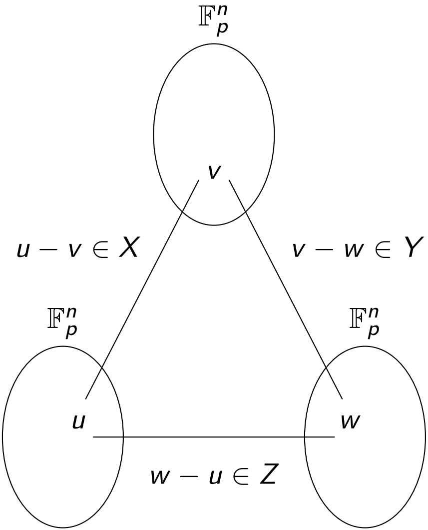

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Remove x from X, Y, or Z if at least $N/3$ edges corresponding to it are removed.

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We can take $\delta = (\varepsilon/3)^{C_p + o(1)}$, where $C_p = 1 + 1/c_p$ is a computable number. The exponent C_p is sharp.

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In particular, $C_2 = 1 + 1/(5/3 - \log_2 3) \approx 13.239$ and $C_3 = 1 + 1/c_3$ where $c_3 = 1 - \frac{\log b}{\log 3}$, $b = a^{-2/3} + a^{1/3} + a^{4/3}$, and $a = \frac{\sqrt{33}-1}{8}$, so $C_3 \approx 13.901$.

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In particular, $C_2 = 1 + 1/(5/3 - \log_2 3) \approx 13.239$ and $C_3 = 1 + 1/c_3$ where $c_3 = 1 - \frac{\log b}{\log 3}$, $b = a^{-2/3} + a^{1/3} + a^{4/3}$, and $a = \frac{\sqrt{33}-1}{8}$, so $C_3 \approx 13.901$.

Removal lemma proof sketch

Theorem (F., L. M. Lovász)

With $\delta = (\varepsilon/3)^{C_p}$, if $X, Y, Z \subset \mathbb{F}_p^n$ have at most δp^{2n} triangles in $X \times Y \times Z$, then we can delete εp^n points and remove all triangles.

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Goal 3

With $\delta = \varepsilon^{C_p + o(1)}$, if we have εN disjoint rainbow triangles with each element in $\approx \beta N$ rainbow triangles, then $\beta \geq \delta/\varepsilon$.

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From the multicolor sum-free theorem

$$\varepsilon \ll |S|^{-c_p} \approx (1/\beta)^{-c_p},$$

which gives $\delta \leq \varepsilon^{C_p + o(1)}$.

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Theorem (Green)

$\forall\ \varepsilon > 0$ there is a least $n(\varepsilon)$ such that if $n \geq n(\varepsilon)$, then $\forall\ A \subset \mathbb{F}_3^n$ of density α, there is a nonzero d such that the density of 3-term arithmetic progressions with common difference d is at least $\alpha^3 - \varepsilon$.

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Theorem* (F.-Pham-Zhao)

A similar result holds in abelian groups and in $[N]$.

Half the random bound

Definition

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Fix $p \geq 47$ a prime.

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For $\beta < \alpha^3 \leq 1/8$, let $n = n_p(\alpha, \beta)$ be the least integer such that for every $A \subset \mathbb{F}_p^n$ of density α, there is a nonzero d such that the density of 3-term APs with common difference d is at least β.

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Corollary (F.-Pham)

If $\alpha \leq 1/2$, then $n(\alpha, \alpha^c)$ is a tower of $1/\alpha$ of height $\Theta(\log(1/c))$.

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Question:

Asymptotics?

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Extend the new cap set theorem to longer arithmetic progressions.

Thank you!