Extremal Problems for Spaces of Matrices

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Plan

Maximal Rank and Matching Numbers

- Flanders theorem and its extensions
- Maximal rank in the exterior algebra
- Edmonds and Lovász min-max theorems

Minimal Rank: between Algebra and Topology

- Algebraically closed vs. finite field cases
- Nonsingular spaces of real matrices

Linear Spaces in Nilpotent varieties

- The matrix case: Gerstenhaber theorems
- Some Lie algebra generalizations

Notations

$M_{m \times n}(\mathbb{F})$ - the space of $m \times n$ matrices over a field $\mathbb{F}$.
$M_n(\mathbb{F}) = M_{n \times n}(\mathbb{F})$.

$$\mathrm{Sym}_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : A = A^T\},$$
$$\mathrm{Alt}_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : A = -A^T \text{ \& } A(i,i) = 0 \text{ for all } i\}.$$

For $u \in \mathbb{F}^m, v \in \mathbb{F}^n$ let

$$u \otimes v = u \cdot v^T \in M_{m \times n}(\mathbb{F}).$$

For $S \subset M_{m \times n}(\mathbb{F})$ let:

$$\overline{\rho}(S) = \max\{\mathrm{rk}(A) : A \in S\},$$
$$\underline{\rho}(S) = \min\{\mathrm{rk}(A) : 0 \neq A \in S\}.$$

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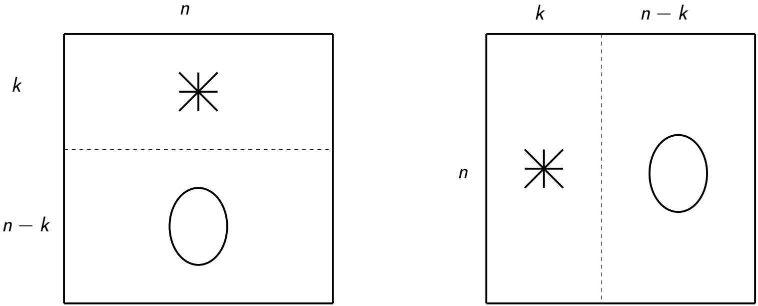
Subspaces of $M_n(\mathbb{F})$ with bounded $\overline{\rho}$

Theorem [Flanders ($|\mathbb{F}| \geq n$), M (any $\mathbb{F}$)]

Let $S \subset M_n(\mathbb{F})$ be a linear subspace such that $\overline{\rho}(S) \leq k$. Then:

- $\dim S \leq kn$.
- $\dim S = kn$ iff either $S = V \otimes \mathbb{F}^n$ or $S = \mathbb{F}^n \otimes V$ for some k-dimensional linear subspace $V \subset \mathbb{F}^n$.

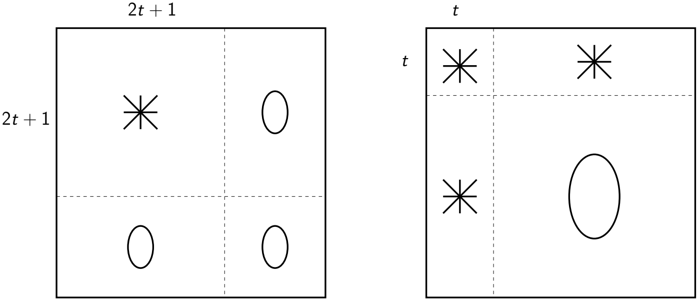


Subspaces of $\mathrm{Alt}_n(\mathbb{F})$ with bounded $\overline{\rho}$

Theorem [M ($|\mathbb{F}| \geq n$), de Seguins Pazzis (any $\mathbb{F}$)]

Let $S \subset \mathrm{Alt}_n(\mathbb{F})$ be a linear subspace such that $\overline{\rho}(S) \leq k = 2t$. Then:

$$\dim S \leq \max\left\{\binom{2t+1}{2}, tn - \binom{t+1}{2}\right\}.$$



$\dim S = \binom{2t+1}{2}$ $\qquad$ $\dim S = tn - \binom{t+1}{2}$

An Extremal Problem for Graph Matchings

A Matching in a graph is a family of pairwise disjoint edges.
The Matching Number of $G = (V, E)$:

$$\nu(G) = \max\{|M| : M \subset E \text{ is a matching }\}.$$

Theorem [Erdős-Gallai]: If $G = (V, E)$ satisfies $\nu(G) \leq t$ then

$$|E| \leq \max\left\{ \binom{2t+1}{2}, t|V| - \binom{t+1}{2} \right\}.$$

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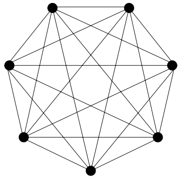
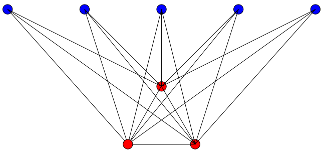


$K_t \vee (n-t)K_1$



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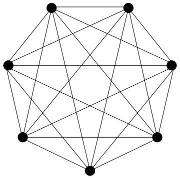

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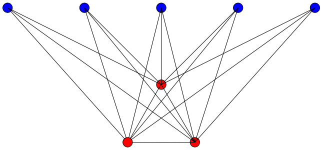

Proof Idea

Let $S \subset \mathrm{Alt}_n(\mathbb{F})$ such that $\overline{\rho}(S) \leq k = 2t$.

- Associate with S a graph $G_S = ([n], E_S)$, with $|E_S| = \dim S$.
- Main point: The matching number of G_S satisfies

$$\nu(G_S) \leq \frac{\overline{\rho}(S)}{2}.$$

- Use extremal graph theory to bound from above $\dim S = |E_S|$.

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From Matrix Spaces to Graphs

The Colexicographic Order

$[n] = \{1, \ldots, n\}$, $[n]^2_< = \{(i, j) \in [n]^2 : i < j\}$.

$$(i,j) \prec (i',j') \quad \Leftrightarrow \quad j < j' \text{ or } (j = j' \;\; \& \;\; i < i').$$

The Leading Entry of a Matrix

For $0 \neq A = \big(A(i,j)\big)_{i,j=1}^{n} \in \mathrm{Alt}_n(\mathbb{F})$ let

$$q(A) = \max\{(i,j) \in [n]^2_< : A(i,j) \neq 0\}.$$

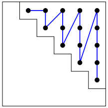
The Graph of an Alternating Space

For a subspace $S \subset \mathrm{Alt}_n(\mathbb{F})$ let $G_S = ([n], E_S)$, where

$$E_S = \big\{\{i,j\} : (i,j) = q(A) \text{ for some } 0 \neq A \in S\big\}.$$

Examples

The Colexicographic Order:



The Leading Entry of a Matrix

$$A = \begin{bmatrix} 0 & * & * & * & 0 \\ * & 0 & * & 1 & 0 \\ * & * & 0 & 0 & 0 \\ * & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad q(A) = (2, 4).$$

The Graph of an Alternating Space

$$S = \left\{ \begin{bmatrix} 0 & x & 0 \\ -x & 0 & y \\ 0 & -y & 0 \end{bmatrix} : x, y \in \mathbb{F} \right\}$$

$G_S =$

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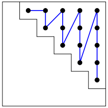

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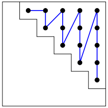

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$$A = \begin{bmatrix} 0 & * & * & * & 0 \\ * & 0 & * & \textcircled{1} & 0 \\ * & * & 0 & 0 & 0 \\ * & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad q(A) = (2,4).$$

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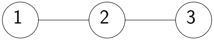

The Pfaffian of an Alternating Matrix

$C = \big(C(i,j)\big)_{i,j=1}^{n} \in \mathrm{Alt}_n(\mathbb{F})$ of even order $n = 2t$.
$\mathcal{M}_n$ - all perfect matchings in K_n. For

$$M = \big\{\{k_1 < \ell_1\}, \dots, \{k_t < \ell_t\}\big\} \in \mathcal{M}_n$$

let

$$\theta(M) = \mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ k_1 & \ell_1 & \cdots & k_t & \ell_t \end{pmatrix}.$$

The Pfaffian of C is:

$$\mathrm{Pf}(C) = \sum_{M \in \mathcal{M}_n} \theta(M) \prod_{i=1}^{t} C(k_i, \ell_i).$$

Fact: $\det(C) = \mathrm{Pf}(C)^2$.

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Combinatorial Nullstellensatz

$g(x_1,\ldots,x_t) = \sum_{(\alpha_1,\ldots,\alpha_t)} c(\alpha_1,\ldots,\alpha_t) x_1^{\alpha_1}\cdots x_t^{\alpha_t} \in \mathbb{F}[x_1,\ldots,x_t]$.
$\deg(g) = \max\{\sum_{i=1}^t \alpha_i : c(\alpha_1,\ldots,\alpha_t) \neq 0\}$.

Theorem [Alon]

Assume:

$$\deg(g) = \sum_{i=1}^{t} d_i \quad \& \quad c(d_1,\ldots,d_t) \neq 0.$$

Let $\Lambda_1,\ldots,\Lambda_t \subset \mathbb{F}$ such that $|\Lambda_i| > d_i$.
Then there exist $\lambda_1 \in \Lambda_1,\ldots,\lambda_t \in \Lambda_t$ such that

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Main Step

Proposition

Let $n = 2m$ and $B_1, \ldots, B_m \in \mathrm{Alt}_n(\mathbb{F})$.
If $\{q(B_1), \ldots, q(B_m)\}$ is a perfect matching in K_n, then:

$$\overline{\rho}\left(\langle B_1, \ldots, B_m \rangle\right) = n.$$

Sketch of Proof: It can be shown that the monomial $x_1 \cdots x_m$ appears in

$$f(x_1, \ldots, x_m) = \mathrm{Pf}\left(\sum_{i=1}^{m} x_i B_i\right)$$

with a nonzero coefficient. By the Combinatorial Nullstellensatz there exists a $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{F}^m$ such that

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Subspaces $\mathrm{Sym}_n(\mathbb{F})$ with bounded $\overline{\rho}$

Theorem [M ($|\mathbb{F}| \geq n$), de Seguins Pazzis (any $\mathbb{F}$)]
Let $S \subset \mathrm{Sym}_n(\mathbb{F})$ be a linear subspace such that $\overline{\rho}(S) \leq k$.

- If $k = 2t$ then

$$\dim S \leq \max\left\{\binom{2t+1}{2}, tn - \binom{t}{2}\right\}.$$

- If $k = 2t+1$ then

$$\dim S \leq \max\left\{\binom{2t+2}{2}, tn - \binom{t}{2} + 1\right\}.$$

Exterior Powers and the Plücker Embedding

The p-th Exterior Power

$\mathtt{I}_p(V)$= subspace of $V^{\otimes p}$ generated by

$$v_1 \otimes \cdots \otimes v_p - \mathrm{sgn}(\pi) v_{\pi(1)} \otimes \cdots \otimes v_{\pi(p)}$$

where $v_1, \ldots, v_p \in V$, $\pi \in \mathbb{S}_p$.

$$\wedge^p V = V^{\otimes p} / \mathtt{I}_p(V).$$

The Plücker Embedding

$G_p(V)$ - the Grassmannian of p-dimensional linear subspaces of V, embeds into $\mathbb{P}(\wedge^p V)$ by the map

$$\langle u_1, \ldots, u_p \rangle \to [u_1 \wedge \cdots \wedge u_p].$$

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Maximal Linear Subspaces of $G_p(V)$

Classical Fact

Let M be a maximal linear space in $G_p(V) \subset \mathbb{P}(\wedge^p V)$. Then either:

$$M = \{\Lambda \in G_p(V) \ : \ \Lambda \subset U\} \quad \text{for some } U \in G_{p+1}(V),$$

or

$$M = \{\Lambda \in G_p(V) \ : \ \Lambda \supset U'\} \quad \text{for some } U' \in G_{p-1}(V).$$

Equivalent Formulation: If $W \subset \wedge^p V$ is a linear space of decomposable p-vectors then either
(i) $W \subset \wedge^p U$ for some $U \in G_{p+1}(V)$, or
(ii) $W \subset \{\alpha \wedge v \ : \ v \in V\}$ for some $\alpha = v_1 \wedge \cdots \wedge v_{p-1} \in \wedge^p V$.
In particular:

$$\dim W \leq \max\{p+1, n-p+1\}.$$

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The Rank of a p-Vector

Enveloping Space of $w \in \wedge^p V$:

$$E(w) = \bigcap \{U \subset V : w \in \wedge^p U\}.$$

Rank of w:

$$\text{rk}(w) = \dim E(w).$$

Example:

$$\begin{aligned} w &= e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_2 \wedge e_4 + e_3 \wedge e_5 \wedge e_6 + e_4 \wedge e_5 \wedge e_6 \\ &= e_1 \wedge e_2 \wedge (e_3 + e_4) + (e_3 + e_4) \wedge e_5 \wedge e_6. \end{aligned}$$

$$E(w) = \langle e_1, e_2, e_3 + e_4, e_5, e_6 \rangle \quad , \quad \text{rk}(w) = 5.$$

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Subspaces of $\wedge^p V$ of Bounded Rank

For $p \leq k \leq n = \dim V$ let

$$\epsilon_p(k) = \begin{cases} 1 & p = k \text{ or } p = 2|k, \\ 0 & \text{otherwise.} \end{cases}$$

$m_p(k) =$ the unique m such that

$$\binom{m-1}{p-1} + m \leq k \leq \binom{m}{p-1} + m.$$

Theorem [Gelbord,M]:

If $W \subset \wedge^p V$ satisfies $\overline{\rho}(W) \leq k$, then:

$$\dim W \leq \max\left\{\binom{k+\epsilon_p(k)}{p}, \binom{m_p(k)}{p} + (k - m_p(k))(n - m_p(k))\right\}.$$

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Large Spaces $W \subset \wedge^p V$ with $\overline{\rho}(W) \leq k$

Fix $U \in G_{k+\epsilon_p(k)}(V)$ and let

$$W_1(n,p,k) = \wedge^p U.$$

Then: $\overline{\rho}(W_1) = k$ and $\dim W_1 = \binom{k+\epsilon_p(k)}{p}$.

Fix $U \in G_m(V)$ and $k-m$ decomposable elements $z_1, \ldots, z_{k-m} \in \wedge^{p-1}(U)$, and let

$$W_2(n,p,k) = \wedge^p U + \langle z_1, \ldots, z_{k-m} \rangle \wedge V.$$

Then: $\overline{\rho}(W_2) = k$ and $\dim W_2 = \binom{m_p(k)}{p} + (k - m_p(k))(n - m_p(k))$.

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The PIT Problem

Given $A_1, \ldots, A_m \in M_n(\mathbb{F})$, can $\overline{\rho}(\langle A_1, \ldots, A_m \rangle)$ be computed in deterministic polynomial time?

Remarks

- The problem admits a simple probabilistic solution: Choose random $x_1, \ldots, x_m \in \mathbb{F}$. Then with high probability

$$\text{rk}(\sum_{i=1}^{m} x_i A_i) = \overline{\rho}(\langle A_1, \ldots, A_m \rangle).$$

- There are (highly nontrivial) deterministic algorithms for PIT, when all A_i's are of rank 1, or skew-symmetric of rank 2.

Spaces Generated by Rank 1 Matrices

Let $u_1, \cdots, u_t \in \mathbb{F}^m$, $v_1, \ldots, v_t \in \mathbb{F}^n$ and let

$$S = \langle u_1 \otimes v_1, \ldots, u_t \otimes v_t \rangle \subset M_{m \times n}(\mathbb{F}).$$

Matroid Intersection Theorem [Edmonds]

$$\overline{\rho}(S) = \min_{\mathtt{I} \subset [t]} \left(\dim \langle u_i : i \in \mathtt{I} \rangle + \dim \langle v_j : j \notin \mathtt{I} \rangle \right).$$

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Spaces Generated by Alternating Decomposables

For $u, v \in \mathbb{F}^n$ let $u \wedge v = u \otimes v - v \otimes u \in \mathrm{Alt}_n(\mathbb{F})$.
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A Dual Problem and Graph Rigidity

Let $u_1, v_1, \ldots, u_t, v_t \in \mathbb{F}^n$ and let

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Weak Duality

Switching Roles

Let U, V be linear spaces over $\mathbb{F}$ and let $S \subset \mathrm{Hom}(U, V)$.
View U as a subspace of $\mathrm{Hom}(S, V)$, by $u(s) = s(u)$.

Theorem [M,Šemrl]

- $\underline{\rho}(S) \leq \overline{\rho}(U)$.
- Let $\mathbb{F}$ be infinite. If $\overline{\rho}(U) \leq \dim S - 1$ then:

  either
  $$\overline{\rho}(S) \geq \dim S$$
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Maximal Singular Spaces

Rank 1 Generation

The minimal dimension of a maximal singular subspace $W \subset M_n(\mathbb{C})$ generated by rank 1 matrices is $\lfloor \frac{3n^2-2n}{4} \rfloor$.

Example: $W = U_1 \otimes \mathbb{F}^n + \mathbb{F}_n \otimes U_2$ where $\dim U_1 = \frac{n}{2}, \dim U_2 = \frac{n}{2} - 1$.

Alternating Rank 2 Generation

The minimal dimension of a maximal singular subspace $W \subset \wedge^2 \mathbb{C}^n$ generated by decomposable elements is $\frac{3n}{2} - 3$.

Example: $W = \bigoplus_{i=1}^{\frac{n}{2}-1} \wedge^2 U_i$ where the U_i's are 3-dimensional spaces in general position.

Theorem [Draisma]

For infinitely many n's there exist 8-dimensional maximal singular spaces in $M_n(\mathbb{C})$.

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Subspaces of $M_n(\mathbb{F})$ with Bounded $\underline{\rho}$

Rank Varieties

$$R_{n,k}(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \mathrm{rk}(A) \leq k\}.$$

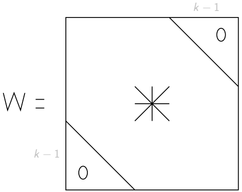
$$f_{\mathbb{F}}(n,k) = \max\{\dim S : S \subset M_n(\mathbb{F}), \underline{\rho}(S) \geq k\}$$

$$= \max\{\dim S : S \cap R_{n,k-1}(\mathbb{F}) = \{0\}.$$

If $\mathbb{F}$ is algebraically closed then $R_{n,k}(\mathbb{F})$ is an irreducible $(2nk - k^2)$-dimensional affine variety. Hence:

$$f_{\mathbb{F}}(n,k) \leq \mathrm{codim}\, R_{n,k-1}(\mathbb{F}) = (n-k+1)^2.$$

Example:
There exists an $(n-k+1)^2$-dim subspace $S \subset W$ that satisfies $\underline{\rho}(S) = k$.



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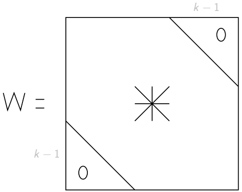

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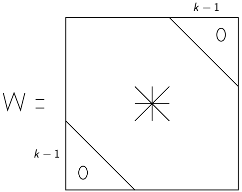

Bounded $\underline{\rho}$: the Finite Field Case

Theorem [Roth]

$$f_{\mathbb{F}_q}(n,k) = n(n-k+1).$$

A Simple Construction [M]:

$$W_q(n,k) = \left\{ g(x) = \sum_{j=0}^{n-k} a_j x^{q^j} \,:\, a_j \in \mathbb{F}_{q^n} \right\} \subset \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong M_n(\mathbb{F}_q).$$

$$\begin{aligned} |W_q(n,k)| = q^{n(n-k+1)} \quad &\Rightarrow \quad \dim_{\mathbb{F}_q} W_q(n,k) = n(n-k+1). \\ 0 \neq g(x) \in W_q(n,k) \quad &\Rightarrow \quad |\ker(g)| \leq q^{n-k} \quad \Rightarrow \\ \dim \ker(g) \leq n-k \quad &\Rightarrow \quad \mathrm{rk}(g) \geq k. \end{aligned}$$

Spaces of Non-Singular Real Matrices

Hurwitz-Radon Number

Write $n = (2\alpha - 1)2^{\gamma+4\delta}$ where $0 \leq \gamma \leq 3$, and let

$$HR(n) = 2^{\gamma} + 8\delta.$$

Theorem [Hurwitz-Radon, Adams]

$$f_{\mathbb{R}}(n, n) = HR(n).$$

Comments

- The lower bound $f_{\mathbb{R}}(n, n) \geq HR(n)$ follows from a construction of Hurwitz and Radon, using Clifford algebras.
- The upper bound $f_{\mathbb{R}}(n, n) \leq HR(n)$ follows from Adams' theorem on the maximal number of vector fields on the sphere.

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The Clifford Algebra $\mathcal{C}_t$

$\mathcal{C}_t = \mathbb{R}$-algebra generated by $e_1, \ldots, e_t$ modulo the relations

$$e_i^2 = -1 \quad \& \quad e_i e_j = -e_j e_i \text{ for } i \neq j.$$

$\mathcal{C}_t$ is 2^t-dimensional algebra with basis:

$$\bigcup_{k=0}^{t} \{e_{i_1} \cdots e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq t\}.$$

Periodicity:

$$\mathcal{C}_{t+8} \cong M_{16}(\mathcal{C}_t).$$

$\mathcal{C}_0$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$

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Nonsingular Spaces via Clifford Algebras

A Representation of $\mathcal{C}_t$ on an n-dimensional real vector space W is an algebra homomorphism $\rho : \mathcal{C}_t \to \mathrm{End}(W)$.

Claim:

$$S = \left\{ \sum_{i=1}^{t} x_i \rho(e_i) : x_i \in \mathbb{R} \right\}$$

is a t-dimensional nonsingular space in $\mathrm{End}(W)$.

Proof: Let $0 \neq w \in W$. If $\sum_{i=1}^{t} x_i \rho(e_i)(w) = 0$ then

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Fix $n \geq 1$. Using the classification of Clifford algebras, it can be checked that for $t = HR(n)$, there is a representation of C_t on $\mathbb{R}^n$. Therefore:

$$f_{\mathbb{R}}(n, n) \geq HR(n).$$

Examples

$n = 2$

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$n = 4$

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Adams Theorem

Vector fields on the $(n-1)$-Sphere

$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

A vector field on S^{n-1} is a continuous map

$$\phi : S^{n-1} \to \mathbb{R}^n$$

such that for all $x \in S^{n-1}$:

$$\phi(x) \cdot x = 0.$$

Theorem [Adams]: Let $\phi_1, \ldots, \phi_k$ be vector fields on S^{n-1} such that $\phi_1(x), \ldots, \phi_k(x)$ are linearly independent for all $x \in S^{n-1}$. Then:

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K-Theory of the Real Projective Space

Notation

ϵ - the trivial line bundle.

η_{r-1} - the tautological line bundle over $\mathbb{RP}^{r-1}$:

$$E(\eta_{r-1}) = \left\{([x], v) : [x] \in \mathbb{RP}^{r-1}, v \in \langle x \rangle\right\}.$$

Theorem [Adams]

$$\widetilde{K}_{\mathbb{R}}(\mathbb{RP}^{r-1}) \cong \mathbb{Z}/(2^{\lambda(r-1)})$$

where

$$\lambda(r-1) = |\{1 \leq k \leq r-1 : k \equiv 0, 1, 2, 4 \pmod 8\}|.$$

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Proof of $f_{\mathbb{R}}(n, n) \leq HR(n)$

Let $S = \langle A_1, \ldots, A_r \rangle$ be a nonsingular subspace of $M_n(\mathbb{R})$.
Suppose for contradiction that $r = HR(n) + 1$.
Define a bundle map

$$g : \eta_{r-1}^{\oplus n} \to \epsilon^{\oplus n}$$

by

$$g\left([x], \theta_1 x, \ldots \theta_n x\right) = \left([x], \left(\sum_{i=1}^{r} x_i A_i \right) \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} \right).$$

The nonsingularity of S implies that g is an isomorphism.
Therefore $n(\eta_{r-1} - \epsilon) = 0$ in $\widetilde{K}_{\mathbb{R}}(\mathbb{RP}^{r-1})$. Hence:

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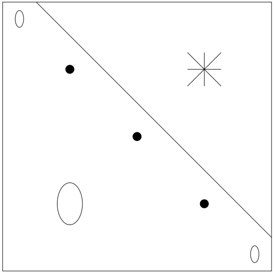
Spaces of Nilpotent Matrices

Theorem [Gerstenhaber ($|\mathbb{F}| \geq n$), Serežkin (any $\mathbb{F}$)]

Let $S \subset M_n(\mathbb{F})$ be a linear space of nilpotent matrices. Then:

- $\dim S \leq \binom{n}{2}$.
- $\dim S = \binom{n}{2}$ iff S is similar to the space of strictly upper diagonal matrices.



Spaces of Symmetric Nilpotent Matrices

Theorem [M,Radwan]

The maximal dimension of a linear subspace of $\mathrm{Sym}_n(\mathbb{C})$ consisting of nilpotent matrices is $\lfloor \frac{n^2}{4} \rfloor$.

Example

For $n = 2m$ let

$$S = \begin{bmatrix} X + X^t + Y & i(X^t - X + Y) \\ i(X - X^t + Y) & X + X^t - Y \end{bmatrix} \subset \mathrm{Sym}_n(\mathbb{C})$$

where $X \in M_m(\mathbb{C})$ is strictly upper triangular and $Y \in \mathrm{Sym}_m(\mathbb{C})$. Then $\dim S = m^2 = \frac{n^2}{4}$ and S is nilpotent.

Nilpotent Subspaces of Lie Algebras

$\mathfrak{g}$ - a complex semi-simple Lie algebra.
$\mathfrak{h}$ - a fixed Cartan subalgebra of $\mathfrak{g}$.
$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ - a fixed Cartan decomposition.

Theorem [Radwan, M]

If $W \subset \mathfrak{g}$ is a linear subspace of ad nilpotent elements then

$$\dim W \leq \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g}) = \dim \mathfrak{n}_+ \ .$$

Theorem [Draisma, Kraft, Kuttler]

$$W \ \text{ nilpotent } \ \& \ \dim W = \dim \mathfrak{n}_+ \quad \Rightarrow \quad W \text{ is conjugate to } \mathfrak{n}_+ \ .$$

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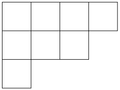
Partitions

A partition of n: $\boldsymbol{p} = (p_1, \ldots, p_t)$ such that

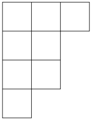
$$p_1 \geq \cdots \geq p_t \geq 0 \quad \& \quad \sum_{i=1}^{t} p_i = n.$$

The conjugate partition: $\boldsymbol{p}^* = \boldsymbol{q} = (q_1, \ldots, q_s)$ where

$$q_i = |\{j : p_j \geq i\}|.$$



$\boldsymbol{p} = (4, 3, 1)$



$\boldsymbol{p}^* = (3, 2, 2, 1)$

Nilpotent Orbits

Jordan Form

$$J_{\mathbf{p}} = \begin{bmatrix} J_{p_1} & 0 & \dots & 0 \\ 0 & J_{p_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{p_t} \end{bmatrix}$$

where J_k is the $k \times k$ Jordan block.

The Orbit of $J_{\mathbf{p}}$

$$\mathcal{O}_{\mathbf{p}} = \{gJ_{\mathbf{p}}g^{-1} : g \in GL_n(\mathbb{C})\}.$$

The closure $\overline{\mathcal{O}_{\mathbf{p}}}$ is an algebraic variety of dimension

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Linear Spaces in Nilpotent Varieties

Theorem [Gerstenhaber]

If $S \subset \overline{\mathcal{O}_{\boldsymbol{p}}}$ is a linear space then

$$\dim S \leq \frac{1}{2} \dim \overline{\mathcal{O}_{\boldsymbol{p}}}.$$

Problem

What is the analogues statement for Lie Algebras?

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THANK YOU!