

Representations of the Kauffman Bracket Skein Algebra of a Surface

IAS Member's Seminar
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Quantum topology

In the early 1980s, the Jones polynomial for knots and links in 3-space was introduced. Witten followed quickly with an amplification of the Jones polynomial to 3-dimensional manifolds.

Straightaway, these quantum invariants solved some long-standing problems in low-dimensional topology. However, their definition is seemingly unrelated to any of the classically known topological and geometric constructions. To this day there is still no good answer to the question:

What topological or geometric properties do the Jones and Witten quantum invariants measure?

Quantum topology

The [Kauffman bracket skein algebra of a surface](#) has emerged as a likely point of connection. In particular, it

- generalizes the (Kauffman bracket formulation of) Jones' polynomial invariant for links
- features in the Topological Quantum Field Theory description of Witten's 3-manifold invariant
- relates to hyperbolic geometry of surface, particularly the $SL_2\mathbb{C}$ -character variety

Plan for talk

1. Intro to the Kauffman bracket skein algebra $\mathcal{S}^q(\Sigma)$.
2. How $\mathcal{S}^q(\Sigma)$ is related to $SL_2\mathbb{C}$ -character variety.
3. Representations of $\mathcal{S}^q(\Sigma)$.

This talk describes joint work with Francis Bonahon.

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$$\begin{aligned} \langle \text{crossing} \rangle &= q^{\frac{1}{2}} \langle \text{two arcs} \rangle + q^{-\frac{1}{2}} \langle \text{two arcs} \rangle \\ \langle \text{circle with dot} \rangle &= (-q - q^{-1}) \langle \text{empty circle} \rangle. \end{aligned}$$

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$$\begin{aligned} \langle \text{crossing} \rangle &= q^{\frac{1}{2}} \langle \text{two arcs} \rangle + q^{-\frac{1}{2}} \langle \text{two arcs} \rangle \\ \langle \text{framed unknot} \rangle &= (-q - q^{-1}) \langle \text{empty circle} \rangle. \end{aligned}$$

Convention: Link diagrams are drawn with framing perpendicular to the blackboard.

Example

$$\langle \text{figure-eight} \rangle = q^{\frac{1}{2}} \langle \text{two circles} \rangle + q^{-\frac{1}{2}} \langle \text{hourglass} \rangle$$

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$$\begin{aligned}\langle \text{figure-eight} \rangle &= q^{\frac{1}{2}} \langle \text{two circles} \rangle + q^{-\frac{1}{2}} \langle \text{figure-eight with crossing} \rangle \\ &= q^{\frac{1}{2}} (-q - q^{-1})^2 + q^{-\frac{1}{2}} (-q - q^{-1}) \\ &= -q^{\frac{3}{2}} (-q - q^{-1}),\end{aligned}$$



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This shows that (as framed links)  \neq .

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Observation: The Kauffman bracket can be written as a linear combination of diagrams without crossings, which bound disks in S^3 . This implies $\langle K \rangle \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$.

Kauffman bracket in S^3 is multiplicative

Observation:

$$\langle \text{figure-eight} \cup \text{circle} \rangle = \langle \text{figure-eight} \rangle \cdot \langle \text{circle} \rangle$$

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$$\langle \text{figure-eight} \quad \text{circle} \rangle = \langle \text{figure-eight} \rangle \cdot \langle \text{circle} \rangle$$

||

$$\langle \text{figure-eight over circle} \rangle$$

Juxtaposition/superposition defines a (commutative) multiplication of the Kauffman bracket in S^3 :

$$\langle K_1 \rangle \cdot \langle K_2 \rangle = \langle K_1 \text{ "next to" } K_2 \rangle = \langle K_1 \text{ "over" } K_2 \rangle$$

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with multiplication by juxtaposition/superposition of framed links.

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with multiplication by superposition of framed links.

Example

- \exists loops not bounding disks \implies many generators of $\mathcal{S}^q(\Sigma)$.

$$\langle \text{Diagram 1} \rangle = q^{\frac{1}{2}} \langle \text{Diagram 2} \rangle + q^{-\frac{1}{2}} \langle \text{Diagram 3} \rangle$$

The diagram shows an equation between three terms, each enclosed in angle brackets. Each term consists of a circle containing a genus-1 surface (a torus). A teal-colored loop is drawn on the surface. In the first diagram, the loop is a figure-eight that crosses itself on the torus. In the second diagram, the loop is a single closed curve that does not bound a disk. In the third diagram, the loop is another single closed curve that does not bound a disk. The equation indicates that the first diagram is equal to the sum of the second and third diagrams, weighted by $q^{\frac{1}{2}}$ and $q^{-\frac{1}{2}}$ respectively.

Example

- \exists loops not bounding disks \implies many generators of $\mathcal{S}^q(\Sigma)$.

$$\langle \text{loop} \rangle = q^{\frac{1}{2}} \langle \text{loop} \rangle + q^{-\frac{1}{2}} \langle \text{loop} \rangle$$

The diagram shows a circle containing a teal loop. The equation states that this loop is equal to the sum of two other loops, each scaled by a power of q . The first loop on the right has a different orientation of the teal loop, and the second loop has a different orientation of the teal loop.

- $\text{loop} \neq \text{loop} \implies$ multiplication is not commutative.

$$\langle \text{loop} \rangle \cdot \langle \text{loop} \rangle \neq \langle \text{loop} \rangle \cdot \langle \text{loop} \rangle$$

The diagram shows two equations. The first equation shows the product of two loops (one teal, one black) resulting in a loop with a different orientation. The second equation shows the product of two loops (one black, one teal) resulting in a loop with a different orientation. The two results are shown to be not equal.

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Remarks

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$$\Sigma \longmapsto V^q(\Sigma)$$

$V^q(\Sigma)$ is a finite dim. quotient of $\mathcal{S}^q(M)$, where $\partial M = \Sigma$.
The action $\mathcal{S}^q(\Sigma) \curvearrowright \mathcal{S}^q(M)$ provides an irreducible repn

$$\rho_{\text{Witten}} : \mathcal{S}^q(\Sigma) \longrightarrow \text{End}(V^q(\Sigma))$$

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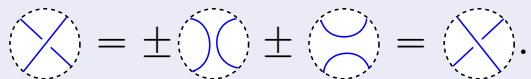
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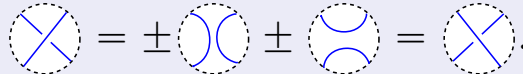
Algebraic Properties of $\mathcal{S}^q(\Sigma)$

1. $\mathcal{S}^q(\Sigma)$ is *not commutative*, except when:

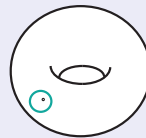
- Σ is a 2-sphere with 0, 1, or 2 punctures, disk with 1 or 2 punctures, or annulus
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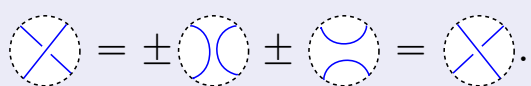
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2. $Z(\mathcal{S}^q(\Sigma))$ contains puncture loops.

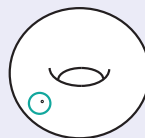


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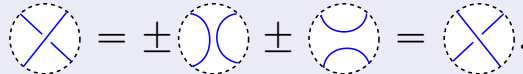
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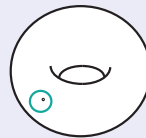
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3. (Przytycki, Turaev, 1990) $\mathcal{S}^q(\Sigma)$ is (usually) *infinitely generated as a module*, by framed links whose projection to Σ has no crossings.

4. (Bullock, 1999) $\mathcal{S}^q(\Sigma)$ is *finitely generated as an algebra*.

Algebraic Properties of $\mathcal{S}^q(\Sigma)$ (very recently proved)

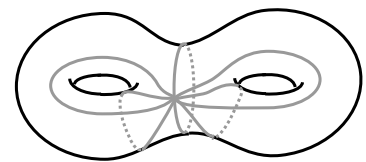
1. (Przytycki-Sikora, Le, Bonahon-W.) $\mathcal{S}^q(\Sigma)$ has *no zero divisors*.
2. (Przytycki-Sikora) $\mathcal{S}^q(\Sigma)$ is *Noetherian*.
3. (Frohman-Kania-Bartoszyńska-Le) $\mathcal{S}^q(\Sigma)$ is *almost Azyumaya*.
4. (Bonahon-W., Le, Frohman-Kania-Bartoszyńska-Le)
Its center $Z(\mathcal{S}^q(\Sigma))$ is *determined*.

Part II

How $\mathcal{S}^q(\Sigma)$ is related to hyperbolic geometry

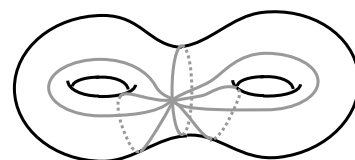
Hyperbolic geometry

Let Σ be a surface with finite topological type.

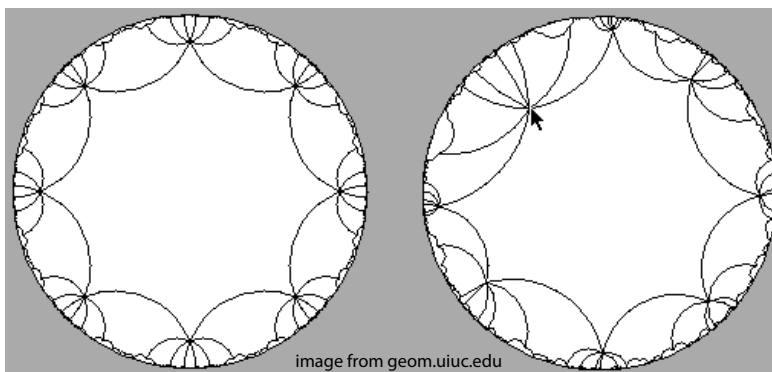


Hyperbolic geometry

Let Σ be a surface with finite topological type.



Its **Teichmüller space** $\mathcal{T}(\Sigma)$ consists of isotopy classes of complete hyperbolic metrics on Σ with finite area.



Every metric $m \in \mathcal{T}(\Sigma)$ corresponds to a **monodromy representation** $r_m : \pi_1(\Sigma) \rightarrow \text{Isom}^+(\mathbb{H}^2) = \text{PSL}_2(\mathbb{R})$, up to conjugation.

Hyperbolic geometry

Hence consider

$$\mathcal{R}_{\mathrm{SL}_2\mathbb{R}} = \{r : \pi_1(\Sigma) \rightarrow \mathrm{SL}_2\mathbb{R}\} // \mathrm{SL}_2\mathbb{R}.$$

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Notice that we lifted $\mathrm{PSL}_2\mathbb{R}$ to $\mathrm{SL}_2\mathbb{R}$

Hyperbolic geometry

Hence consider

$$\mathcal{R}_{\mathrm{SL}_2\mathbb{C}} = \{r : \pi_1(\Sigma) \rightarrow \mathrm{SL}_2\mathbb{C}\} // \mathrm{SL}_2\mathbb{C}.$$

Notice that we lifted $\mathrm{PSL}_2\mathbb{R}$ to $\mathrm{SL}_2\mathbb{R}$ and complexify to $\mathrm{SL}_2\mathbb{C}$.

Hyperbolic geometry

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Notice that we lifted $\mathrm{PSL}_2\mathbb{R}$ to $\mathrm{SL}_2\mathbb{R}$ and complexify to $\mathrm{SL}_2\mathbb{C}$.

The $\mathrm{SL}_2\mathbb{C}$ -character variety $\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$ is an algebraic variety that contains a copy of Teichmüller space $\mathcal{T}(\Sigma)$ as a real subvariety.

Note that $[r_1] = [r_2]$ if and only if $\mathrm{trace} \circ r_1 = \mathrm{trace} \circ r_2$.

Hyperbolic geometry

Trace functions

For every loop K in Σ , define the **trace function**

$$\mathrm{Tr}_K : \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma) \rightarrow \mathbb{C} \text{ by } \mathrm{Tr}_K(r) = -\mathrm{trace}(r([K])).$$

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The trace functions generate the function algebra $\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)]$ (consisting of all regular functions $\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma) \rightarrow \mathbb{C}$).

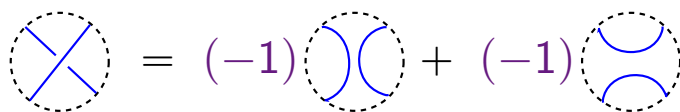
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Important observation: The trace functions Tr_K satisfy the Kauffman bracket skein relation when $q^{\frac{1}{2}} = -1$,



The diagram illustrates the Kauffman bracket skein relation for the trace functions. It shows a circle with a dashed boundary. On the left, the circle contains a blue 'X' formed by two intersecting lines. This is equal to the sum of two terms. The first term is (-1) multiplied by a circle containing two blue arcs that meet at the top and bottom, forming a 'C' shape. The second term is (-1) multiplied by a circle containing two blue arcs that meet at the top and bottom, forming an inverted 'C' shape.

$$\text{Crossing} = (-1) \text{C} + (-1) \text{C'}$$

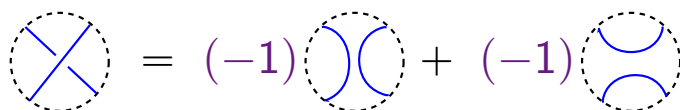
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from identity $\mathrm{Tr}(M)\mathrm{Tr}(N) = \mathrm{Tr}(MN) + \mathrm{Tr}(MN^{-1})$ in $\mathrm{SL}_2\mathbb{C}$.

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$$\text{Crossing} = (-1) \text{Cup} + (-1) \text{Cap} = \text{Crossing}.$$

from identity $\mathrm{Tr}(M)\mathrm{Tr}(N) = \mathrm{Tr}(MN) + \mathrm{Tr}(MN^{-1})$ in $\mathrm{SL}_2\mathbb{C}$.

Skein algebras and hyperbolic geometry

Theorem (... , Bullock-Frohman-Kania-Bartoszyńska, Przytycki-Sikora)

For $q^{\frac{1}{2}} = -1$, there is an *isomorphism* between the skein algebra $\mathcal{S}^1(\Sigma)$ and the function algebra $\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)]$ given by

$$\begin{array}{ccc} [K] & & \mathrm{Tr}_K \\ \mathcal{S}^1(\Sigma) & \longleftrightarrow & \mathbb{C}[\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)]. \end{array}$$

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where $\{ , \}$ is the Weil-Petersson-Atiyah-Bott-Goldman Poisson bracket, using computations of Goldman.

Skein algebra is quantization of Teichmuller space

Theorem (Turaev)

$\mathcal{S}^q(\Sigma)$ is a *quantization* of $\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$ with respect to the Weil-Petersson Poisson structure.

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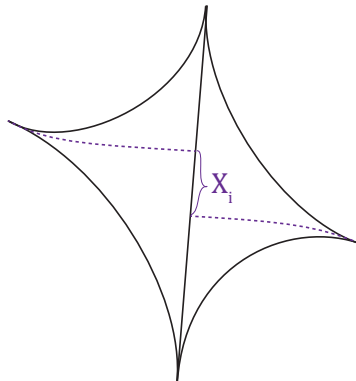
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There's another quantization of Teichmuller space, related to quantum cluster algebras. (Chekhov-Fock, Kashaev, Bonahon et al.)

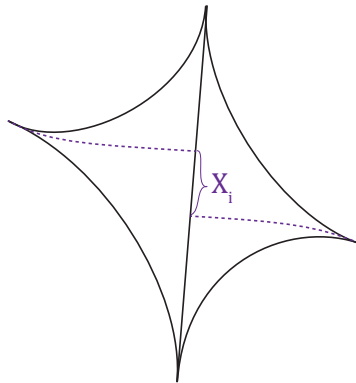
Another quantization of Teichmuller space

Suppose Σ has at least one puncture, with ideal triangulation λ . The **shear parameters** $x_i(r) \in \mathbb{R}$ along edges $\lambda_1, \dots, \lambda_n$ completely describe a hyperbolic metric $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$.



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The trace functions Tr_K are polynomials in $x_i^{\pm\frac{1}{2}}$. So the function algebra $\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)]$ is also generated by **commutative shear parameters** $x_1^{\pm\frac{1}{2}}, \dots, x_n^{\pm\frac{1}{2}}$.

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Following the theory of quantum cluster algebras, Chekhov-Fock define the **quantum Teichmuller space** $\mathcal{T}^q(\Sigma)$ to be the algebra generated by **non-commutative shear parameters** $Z_1^{\pm 1}, \dots, Z_n^{\pm 1}$

- $Z_i Z_j = q^{\pm 1} Z_j Z_i$ if the i th and j th edges share a triangle,
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*There exists an **injective** map $\mathcal{S}^q(\Sigma) \hookrightarrow \mathcal{T}^q(\Sigma)$ which is compatible with $\mathcal{S}^q(\Sigma)$ as a quantization of $\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$ with respect to the Weil-Petersson Poisson structure.*

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Great because multiplication in $\mathcal{T}^q(\Sigma)$ is easier to manipulate.
Not great because the map is complicated.

Summary of Part II

- The Kauffman bracket skein algebra of a surface is the quantization of Teichmuller space (really, $\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)]$) in multiple ways.

- In one

$$\begin{array}{l} \rho : \mathcal{S}^q(\Sigma) \rightarrow \mathbb{C} \quad \in \quad \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma) \\ \rho : \mathcal{S}^q(\Sigma) \rightarrow \mathrm{End}(\mathbb{C}^d) \quad \in \quad \text{quantization of } \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma) \end{array}$$

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What are these reps?

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Part III

Representations of $\mathcal{S}^q(\Sigma)$

Theorem (Bonahon-W.)

Let q be a N -root of 1, with N odd.

Let Σ be a surface with finite topological type.

1. Every irreducible representation ρ of $\mathcal{S}^q(\Sigma)$ uniquely determines
 - *puncture invariants* $p_i \in \mathbb{C}$ for every puncture of Σ
 - *classical shadow* $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$
2. For every compatible $\{p_i\}$ and $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$, there exists an irreducible representation ρ of $\mathcal{S}^q(\Sigma)$ with those invariants.

We'll sketch the methods of construction, starting with a revisit to the algebraic structure of $\mathcal{S}^q(\Sigma)$. This will involve Chebyshev polynomials.

Remarks on Theorem

irred representation
 $\rho : \mathcal{S}^q(\Sigma) \rightarrow \text{End}(\mathbb{C}^n)$

$1-1?$
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irred representation $\rho : \mathcal{S}^g(\Sigma) \rightarrow \text{End}(\mathbb{C}^n)$ $\xleftrightarrow{1-1?}$ puncture invs $p_i \in \mathbb{C}$ and classical shadow $r \in \mathcal{R}_{\text{SL}_2\mathbb{C}}(\Sigma)$

- \exists two irreps of $\mathcal{S}^g(\Sigma)$ (ρ_{Witten} and ρ_{Thm}) whose classical shadow is the trivial character $r : \pi_1(\Sigma) \rightarrow \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

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- **Breaking news:** In July preprint, Frohman, Kania-Bartoszynska, and Le prove 1-1 for a Zariski dense, open subset of $\mathcal{R}_{\text{SL}_2\mathbb{C}}(\Sigma)$.

Chebyshev polynomials

Chebyshev polynomials

Let $T_n(x)$ be the **Chebyshev polynomials of the first kind**, defined recursively as follows:

$$T_0(x) = 2$$

$$T_1(x) = x$$

$$T_2(x) = x^2 - 2$$

$$T_3(x) = x^3 - 3x$$

\vdots

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Note:

- $\cos(n\theta) = \frac{1}{2} T_n(2 \cos \theta)$
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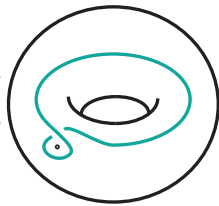
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- Witten's TQFT uses the Chebyshevs of the *second* kind.

Threading skeins by Chebyshev polynomials

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The diagram shows the result of applying the Chebyshev polynomial T_3 to the skein K . The leftmost diagram is a purple loop with a small circle at the bottom, labeled T_3 . This is equal to the middle diagram, which is a teal loop with three turns, minus three times the original teal loop K .

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- **Observe:** For a framed link K with n crossings, the Kauffman bracket $[K]$ has 2^n resolutions, whereas $[K^{T_N}]$ has $\mathcal{O}(2^{n^2})$ resolutions.

Threading skeins by Chebyshev polynomials

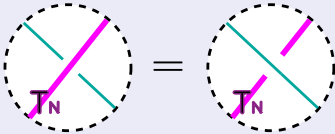
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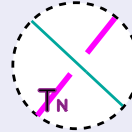
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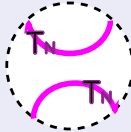
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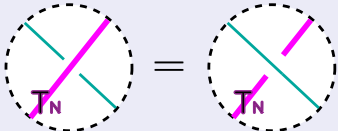
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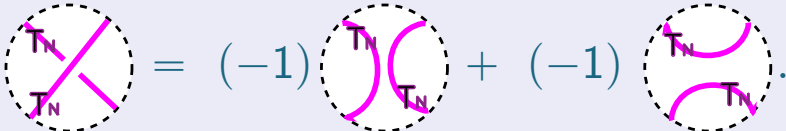
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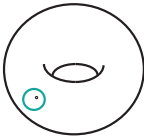
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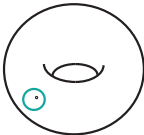
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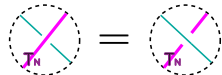
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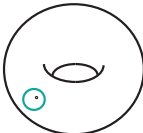
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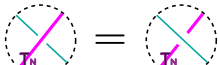
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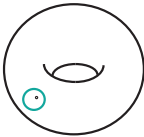
- A threaded link $[K^{T_N}] \in \mathcal{S}^q(\Sigma)$ is central. 

Schur's Lemma $\Rightarrow \rho : [K^{T_N}] \mapsto \hat{\rho}([K]) \cdot \text{Id}$ for $\hat{\rho}([K]) \in \mathbb{C}$.

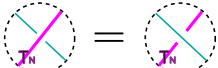
$[K^{T_N}]$ satisfy the skein relation  = -  - .

Sketch proof of Theorem, part 1




Suppose $\rho : \mathcal{S}^q(\Sigma) \rightarrow \text{End}(\mathbb{C}^d)$ is irreducible.

- The i -th puncture loop $[P_i]$ is central. 

Schur's Lemma $\Rightarrow \rho : [P_i] \mapsto p_i \cdot \text{Id}$ for $p_i \in \mathbb{C}$ **puncture invs.**

- A threaded link $[K^{TN}] \in \mathcal{S}^q(\Sigma)$ is central. 

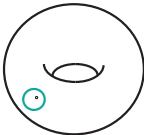
Schur's Lemma $\Rightarrow \rho : [K^{TN}] \mapsto \hat{\rho}([K]) \cdot \text{Id}$ for $\hat{\rho}([K]) \in \mathbb{C}$.

$[K^{TN}]$ satisfy the skein relation  = $-$  $-$ . So

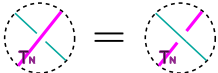
$$\hat{\rho} : \mathcal{S}^1(\Sigma) \longrightarrow \mathbb{C}$$

Sketch proof of Theorem, part 1




Suppose $\rho : \mathcal{S}^q(\Sigma) \rightarrow \text{End}(\mathbb{C}^d)$ is irreducible.

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- A threaded link $[K^{TN}] \in \mathcal{S}^q(\Sigma)$ is central. 

Schur's Lemma $\Rightarrow \rho : [K^{TN}] \mapsto \hat{\rho}([K]) \cdot \text{Id}$ for $\hat{\rho}([K]) \in \mathbb{C}$.

$[K^{TN}]$ satisfy the skein relation  = -  - . So

$$\hat{\rho} : \mathcal{S}^1(\Sigma) \longrightarrow \mathbb{C} \quad \longleftrightarrow \quad r \in \mathcal{R}_{\text{SL}_2\mathbb{C}}(\Sigma)$$

classical shadow

from the isomorphism between $\mathcal{S}^q(\Sigma)$ and $\mathbb{C}[\mathcal{R}_{\text{SL}_2\mathbb{C}}(\Sigma)]$.

Sketch proof of Theorem, part 2

Suppose we are given compatible puncture invariants $p_i \in \mathbb{C}$ and classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$.

Recall that for Σ with at least one puncture, the **quantum Teichmuller space** $\mathcal{T}^q(\Sigma)$ is the algebra generated by $Z_1^{\pm 1}, \dots, Z_n^{\pm 1}$ with $Z_i Z_j = q^{\pm 1} Z_j Z_i$ if the i th and j th edges share a triangle. Bonahon-Liu construct all representations of $\mathcal{T}^q(\Sigma)$, which are classified by compatible numbers $p_i \in \mathbb{C}$ and $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$.

When Σ has at least one puncture, compose

$$\rho : \mathcal{S}^q(\Sigma) \hookrightarrow \mathcal{T}^q(\Sigma) \xrightarrow{p_i, r} \mathrm{End}(\mathbb{C}^d).$$

When Σ has no punctures, drill punctures and show there is an invariant subspace that is blind to punctures.

Further remarks

- In 2016 preprint, Frohman-Kania-Bartoszynska describe another method of constructing representations when Σ has at least one puncture, which matches our construction for generic characters by July 2017 preprint of Frohman-Kania-Bartoszynska-Le.
- $\mathcal{S}^q(\Sigma) \hookrightarrow \mathcal{T}^q(\Sigma)$ is used to prove many of the algebraic properties mentioned earlier (e.g., no zero divisors, determination of center).
- How can we use it to more closely tie together the Jones and Witten quantum invariants with hyperbolic geometry?



Thanks!