Representations of the Kauffman Bracket Skein Algebra of a Surface

IAS Member's Seminar
November 20, 2017

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Quantum topology

In the early 1980s, the Jones polynomial for knots and links in 3-space was introduced. Witten followed quickly with an amplification of the Jones polynomial to 3-dimensional manifolds.

Straightaway, these quantum invariants solved some long-standing problems in low-dimensional topology. However, their definition is seemingly unrelated to any of the classically known topological and geometric constructions. To this day there is still no good answer to the question:

What topological or geometric properties do the Jones and Witten quantum invariants measure?

Quantum topology

The Kauffman bracket skein algebra of a surface has emerged as a likely point of connection. In particular, it

- generalizes the (Kauffman bracket formulation of) Jones' polynomial invariant for links
- features in the Topological Quantum Field Theory description of Witten's 3-manifold invariant
- relates to hyperbolic geometry of surface, particularly the $\mathrm{SL}_2\mathbb{C}$-character variety

Plan for talk

1. Intro to the Kauffman bracket skein algebra $\mathcal{S}^q(\Sigma)$.
2. How $\mathcal{S}^q(\Sigma)$ is related to $\mathrm{SL}_2\mathbb{C}$-character variety.
3. Representations of $\mathcal{S}^q(\Sigma)$.

This talk describes joint work with Francis Bonahon.

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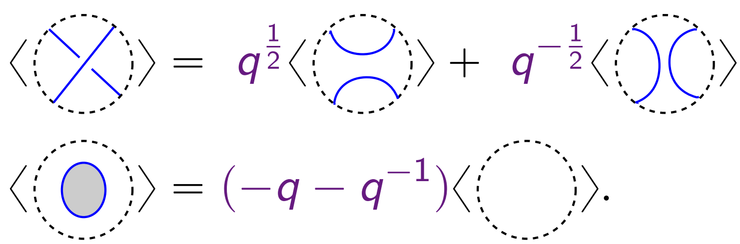

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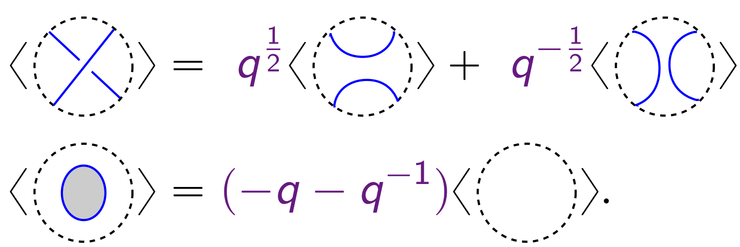
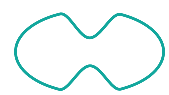


Convention: Link diagrams are drawn with framing perpendicular to the blackboard.

Example

$\langle$  $\rangle$ $=$ $q^{\frac{1}{2}}\langle$  $\rangle + q^{-\frac{1}{2}}\langle$  $\rangle$

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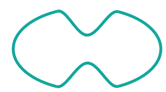
$\langle$  $\rangle = q^{\frac{1}{2}} \langle$  $\rangle + q^{-\frac{1}{2}} \langle$  $\rangle$

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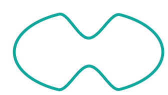
$\langle$  $\rangle = q^{\frac{1}{2}} \langle$  $\rangle + q^{-\frac{1}{2}} \langle$  $\rangle$

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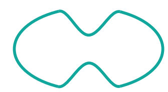
$\langle$  $\rangle = q^{\frac{1}{2}} \langle$  $\rangle + q^{-\frac{1}{2}} \langle$  $\rangle$

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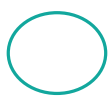
$\langle$  $\rangle = -q - q^{-1}.$

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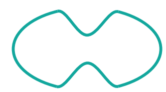
$\langle$  $\rangle$ $= q^{\frac{1}{2}}\langle$  $\rangle + q^{-\frac{1}{2}}\langle$  $\rangle$

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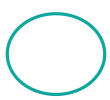
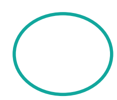
$\langle$  $\rangle$ $= -q - q^{-1}.$

This shows that (as framed links)  $\neq$  .

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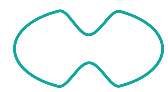
$\langle$  $\rangle = q^{\frac{1}{2}} \langle$  $\rangle + q^{-\frac{1}{2}} \langle$  $\rangle$

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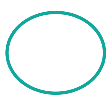
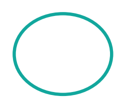
$\langle$  $\rangle = -q - q^{-1}.$

This shows that (as framed links)  $\neq$  .

Observation: The Kauffman bracket can be written as a linear combination of diagrams without crossings, which bound disks in S^3. This implies $\langle K \rangle \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$.

Kauffman bracket in S^3 is multiplicative

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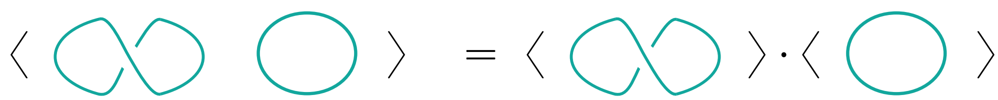

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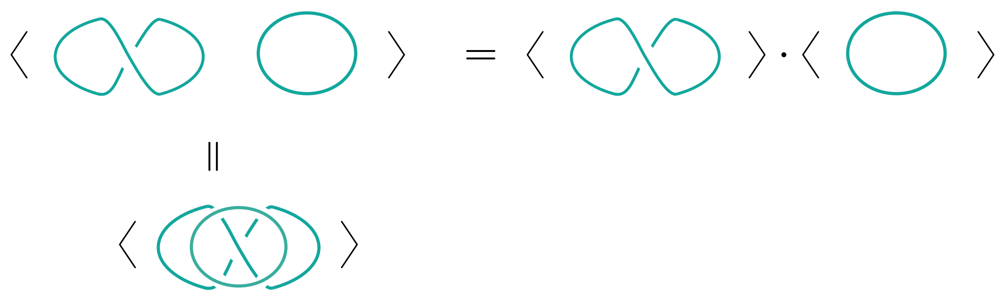

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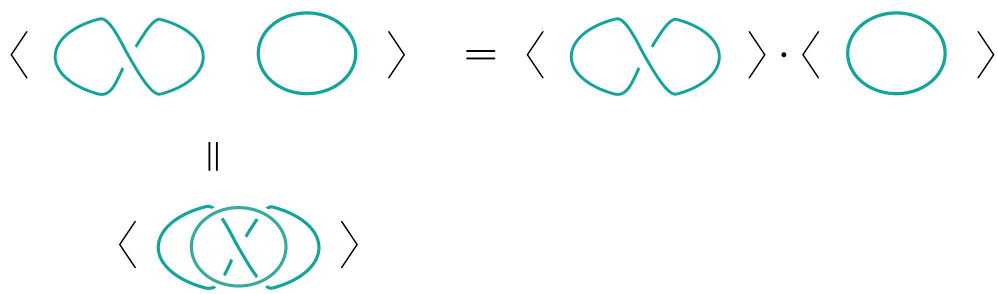

Juxtaposition/superposition defines a (commutative) multiplication of the Kauffman bracket in S^3:

$$\langle K_1 \rangle \cdot \langle K_2 \rangle = \langle K_1 \text{ ``next to'' } K_2 \rangle = \langle K_1 \text{ ``over'' } K_2 \rangle$$

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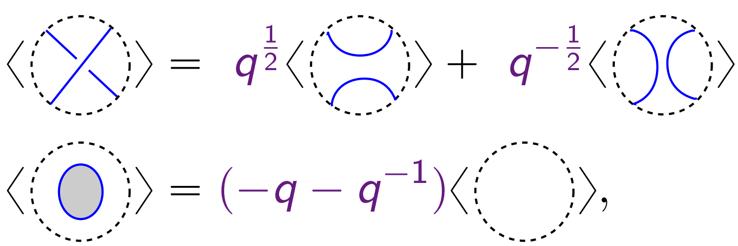

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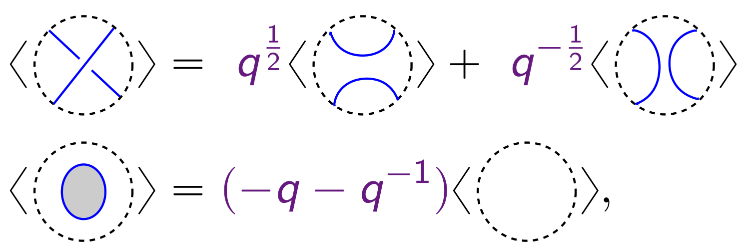

with multiplication by juxtaposition/superposition of framed links.

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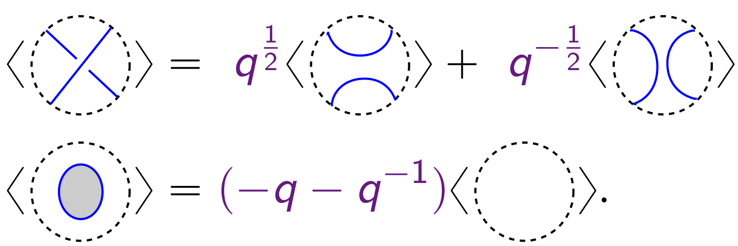

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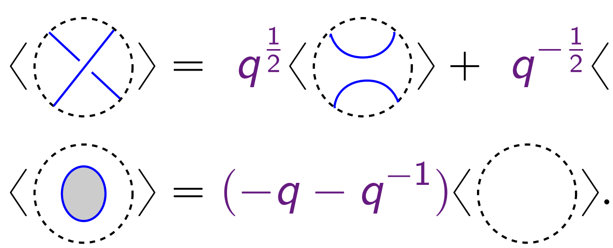
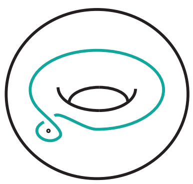
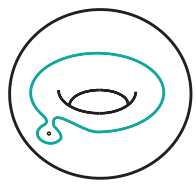
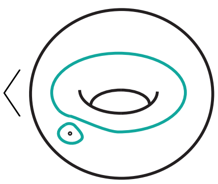


with multiplication by superposition of framed links.

Example

- $\exists$ loops not bounding disks $\implies$ many generators of $\mathcal{S}^q(\Sigma)$.

$\Big\langle$  $\Big\rangle = q^{\frac{1}{2}} \Big\langle$  $\Big\rangle + q^{-\frac{1}{2}} \Big\langle$  $\Big\rangle$

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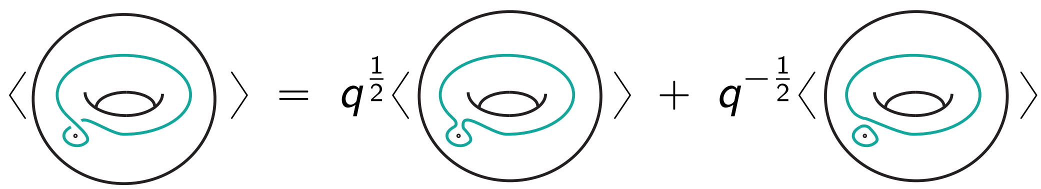
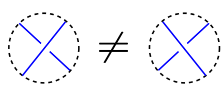
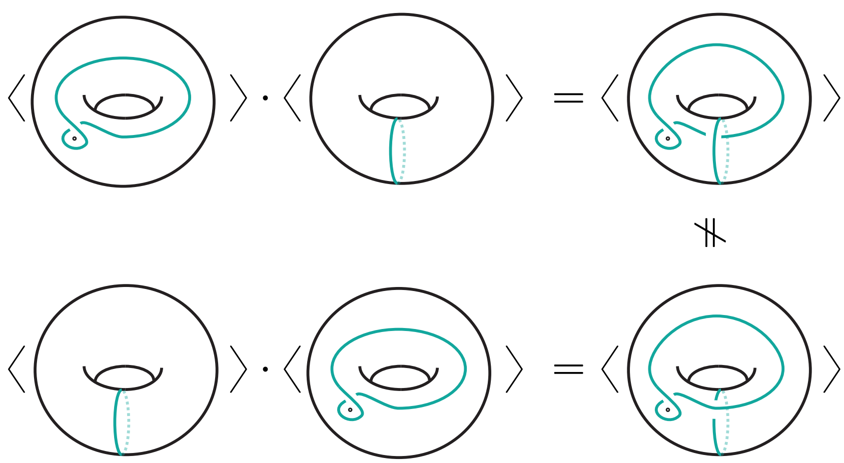


-  $\implies$ multiplication is not commutative.



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$V^q(\Sigma)$ is a finite dim. quotient of $\mathcal{S}^q(M)$, where $\partial M = \Sigma$. The action $\mathcal{S}^q(\Sigma) \circlearrowright \mathcal{S}^q(M)$ provides an irreducible repn

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Algebraic Properties of $\mathcal{S}^q(\Sigma)$

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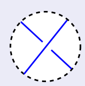
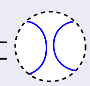
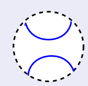
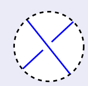
   - $q = 1$, *when* $q^{\frac{1}{2}} = q^{-\frac{1}{2}}$ *and*  $= \pm$  $\pm$  $=$ .

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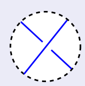
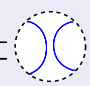
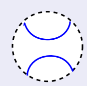
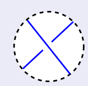
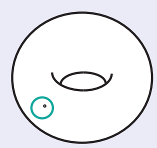
   - $q = 1$, *when* $q^{\frac{1}{2}} = q^{-\frac{1}{2}}$ *and*  $= \pm$  $\pm$  $=$ .

2. $Z(\mathcal{S}^q(\Sigma))$ *contains puncture loops.* 

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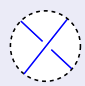
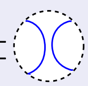
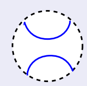
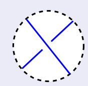
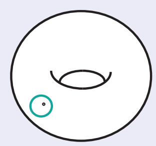
   - *$q = 1$, when $q^{\frac{1}{2}} = q^{-\frac{1}{2}}$ and*  $= \pm$  $\pm$  $=$ .

2. *$Z(\mathcal{S}^q(\Sigma))$ contains puncture loops.* 

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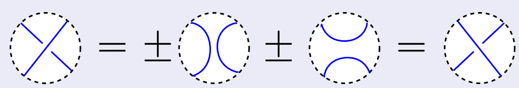
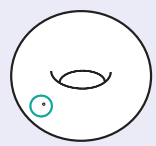
   - *$q = 1$, when $q^{\frac{1}{2}} = q^{-\frac{1}{2}}$ and* .

2. $Z(\mathcal{S}^q(\Sigma))$ *contains puncture loops.* 

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4. *(Bullock, 1999)* $\mathcal{S}^q(\Sigma)$ *is finitely generated as an algebra.*

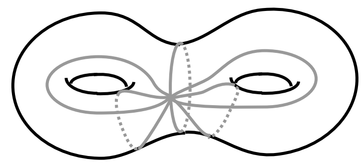
Algebraic Properties of $\mathcal{S}^q(\Sigma)$ (very recently proved)

1. *(Przytycki-Sikora, Le, Bonahon-W.)* $\mathcal{S}^q(\Sigma)$ *has no zero divisors.*
2. *(Przytycki-Sikora)* $\mathcal{S}^q(\Sigma)$ *is Noetherian.*
3. *(Frohman-Kania-Bartoszynska-Le)* $\mathcal{S}^q(\Sigma)$ *is almost Azyumaya.*
4. *(Bonahon-W., Le, Frohman-Kania-Bartozynska-Le)*
   Its center $Z(\mathcal{S}^q(\Sigma))$ *is determined.*

Part II

How $\mathcal{S}^q(\Sigma)$ is related to hyperbolic geometry

Hyperbolic geometry

Let Σ be a surface with finite topological type. 

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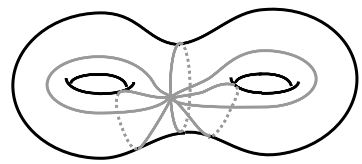
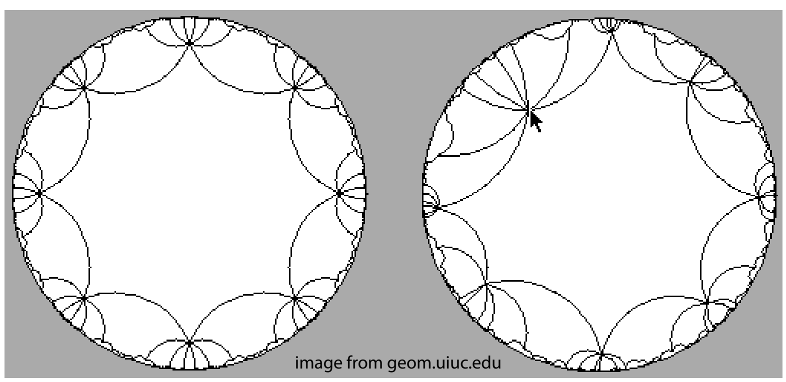
Let Σ be a surface with finite topological type. 

Its Teichmuller space $\mathcal{T}(\Sigma)$ consists of isotopy classes of complete hyperbolic metrics on Σ with finite area.



Every metric $m \in \mathcal{T}(\Sigma)$ corresponds to a monodromy representation $r_m : \pi_1(\Sigma) \to \mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R})$, up to conjugation.

Hyperbolic geometry

Hence consider

$$\mathcal{R}_{\mathrm{SL}_2\mathbb{R}} = \{r : \pi_1(\Sigma) \to \mathrm{SL}_2\mathbb{R}\} /\!\!/ \mathrm{SL}_2\mathbb{R}.$$

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Hence consider

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The $\mathrm{SL}_2\mathbb{C}$-character variety $\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$ is an algebraic variety that contains a copy of Teichmuller space $\mathcal{T}(\Sigma)$ as a real subvariety.

Note that $[r_1] = [r_2]$ if and only if $\mathrm{trace} \circ r_1 = \mathrm{trace} \circ r_2$.

Hyperbolic geometry

Trace functions

For every loop K in Σ, define the trace function

$$\mathrm{Tr}_K : \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma) \to \mathbb{C} \text{ by } \mathrm{Tr}_K(r) = -\mathrm{trace}(r([K])).$$

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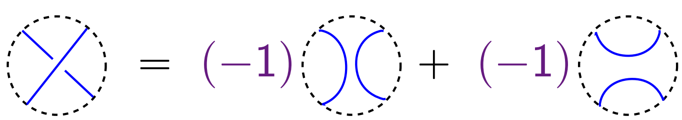
Important observation: The trace functions Tr_K satisfy the Kauffman bracket skein relation when $q^{\frac{1}{2}} = -1$,



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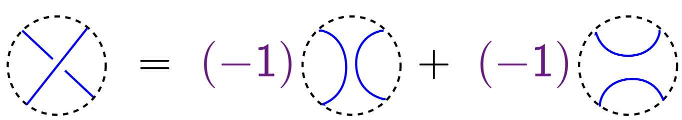

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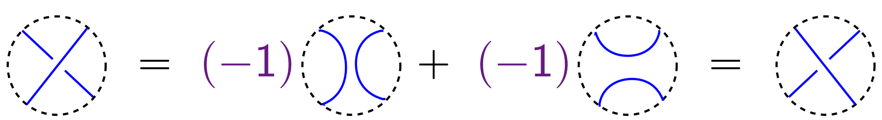

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Theorem (. . . , Bullock-Frohman-Kania-Bartoszynska, Przytycki-Sikora)

For $q^{\frac{1}{2}} = -1$, there is an isomorphism between the skein algebra $\mathcal{S}^1(\Sigma)$ and the function algebra $\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)]$ given by

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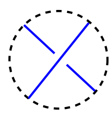
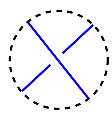
$[K][L] - [L][K] =$  $-$ 

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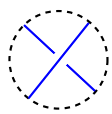
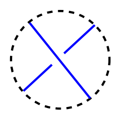
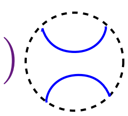
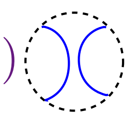
$[K][L] - [L][K] =$  $-$ 

$= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$  $- (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$  .

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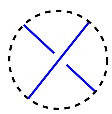
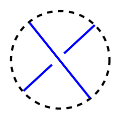
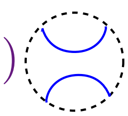
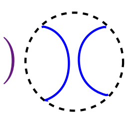
$[K][L] - [L][K] =$  $-$ 

$= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$  $- (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$  .

And when $q = e^{2\pi i \hbar}$,

$$[K][L] - [L][K] = 2\pi i \hbar \left(\qquad\qquad \right) + \text{higher order terms in } \hbar$$

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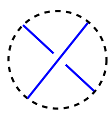
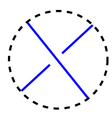
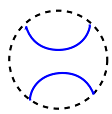
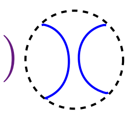
$[K][L] - [L][K] =$  $-$ 

$= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$  $- (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ .

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where $\{\ ,\ \}$ is the Weil-Petersson-Atiyah-Bott-Goldman Poisson bracket, using computations of Goldman.

Skein algebra is quantization of Teichmuller space

Theorem (Turaev)

$\mathcal{S}^q(\Sigma)$ is a quantization of $\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$ with respect to the Weil-Petersson Poisson structure.

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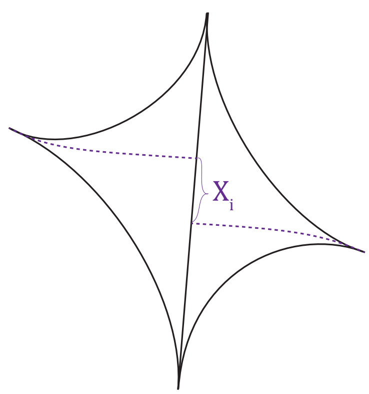
There's another quantization of Teichmuller space, related to quantum cluster algebras. (Chekhov-Fock, Kashaev, Bonahon et al.)

Another quantization of Teichmuller space

Suppose Σ has at least one puncture, with ideal triangulation λ. The shear parameters $x_i(r) \in \mathbb{R}$ along edges $\lambda_1, \ldots \lambda_n$ completely describe a hyperbolic metric $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$.



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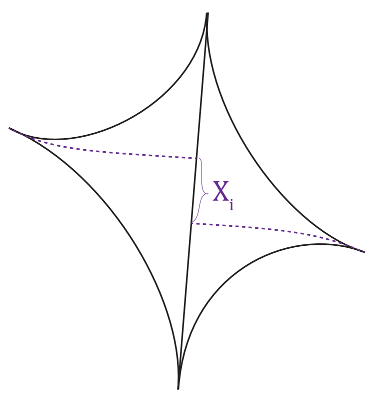

The trace functions Tr_K are polynomials in $x_i^{\pm\frac{1}{2}}$.
So the function algebra $\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)]$ is also generated by commutative shear parameters $x_1^{\pm\frac{1}{2}}, \dots, x_n^{\pm\frac{1}{2}}$.

Another quantization of Teichmuller space

Following the theory of quantum cluster algebras, Chekhov-Fock define the quantum Teichmuller space $\mathcal{T}^q(\Sigma)$ to be the algebra generated by non-commutative shear parameters $Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}$

- $Z_i Z_j = q^{\pm 1} Z_j Z_i$ if the ith and jth edges share a triangle, $Z_i Z_j = Z_j Z_i$ otherwise
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There exists an injective map $\mathcal{S}^q(\Sigma) \hookrightarrow \mathcal{T}^q(\Sigma)$ *which is compatible with* $\mathcal{S}^q(\Sigma)$ *as a quantization of* $\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$ *with respect to the Weil-Petersson Poisson structure.*

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Great because multiplication in $\mathcal{T}^q(\Sigma)$ is easier to manipulate. Not great because the map is complicated.

Summary of Part II

- The Kauffman bracket skein algebra of a surface is the quantization of Teichmuller space (really, $\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)]$) in multiple ways.

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What are these repns?

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Part III

Representations of $\mathcal{S}^q(\Sigma)$

Theorem (Bonahon-W.)

Let q be a N-root of 1, with N odd.
Let Σ be a surface with finite topological type.

1. *Every irreducible representation ρ of $\mathcal{S}^q(\Sigma)$ uniquely determines*
   - *puncture invariants $p_i \in \mathbb{C}$ for every puncture of Σ*
   - *classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$*
2. *For every compatible $\{p_i\}$ and $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$, there exists an irreducible representation ρ of $\mathcal{S}^q(\Sigma)$ with those invariants.*

We'll sketch the methods of construction, starting with a revisit to the algebraic structure of $\mathcal{S}^q(\Sigma)$. This will involve Chebyshev polynomials.

Remarks on Theorem

irred representation $\rho : \mathcal{S}^q(\Sigma) \to \mathrm{End}(\mathbb{C}^n)$ $\overset{1-1?}{\longleftrightarrow}$ puncture invs $p_i \in \mathbb{C}$ and classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$

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- $\exists$ two irrepns of $\mathcal{S}^q(\Sigma)$ (ρ_{Witten} and ρ_{Thm}) whose classical shadow is the trivial character $r : \pi_1(\Sigma) \to \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$.

Remarks on Theorem

irred representation $\rho : \mathcal{S}^q(\Sigma) \to \mathrm{End}(\mathbb{C}^n)$ $\overset{1-1?}{\longleftrightarrow}$ puncture invs $p_i \in \mathbb{C}$ and classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$

- $\exists$ two irrepns of $\mathcal{S}^q(\Sigma)$ (ρ_{Witten} and ρ_{Thm}) whose classical shadow is the trivial character $r : \pi_1(\Sigma) \to \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$.
- Breaking news: In July preprint, Frohman, Kania-Bartoszynska, and Le prove 1-1 for a Zariski dense, open subset of $\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$.

Chebyshev polynomials

Chebyshev polynomials

Let $T_n(x)$ be the Chebyshev polynomials of the first kind, defined recursively as follows:

$$\begin{aligned} T_0(x) &= 2 \\ T_1(x) &= x \\ T_2(x) &= x^2 - 2 \\ T_3(x) &= x^3 - 3x \\ &\vdots \\ T_n(x) &= xT_{n-1}(x) - T_{n-2}(x). \end{aligned}$$

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- $\cos(n\theta) = \frac{1}{2} T_n(2\cos\theta)$
- $\mathrm{Tr}(M^n) = T_n(\,\mathrm{Tr}(M)\,)$ for all $M \in \mathrm{SL}_2(\mathbb{C})$.

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- Witten's TQFT uses the Chebyshevs of the *second* kind.

Threading skeins by Chebyshev polynomials

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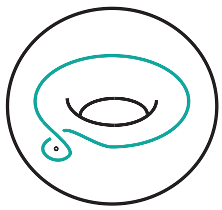
- Example: $[K] = \Big\langle$  $\Big\rangle$ and $T_3 = x^3 - 3x$,

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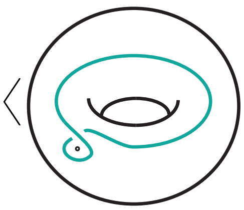
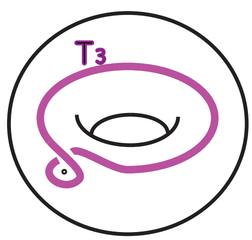
- Example: $[K] = \langle$  $\rangle$ and $T_3 = x^3 - 3x$,

$[K^{T_3}] =$ 

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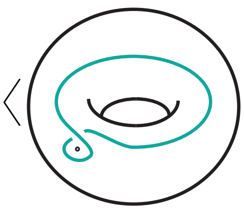
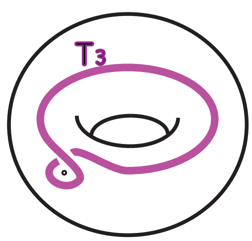
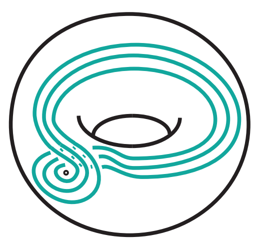
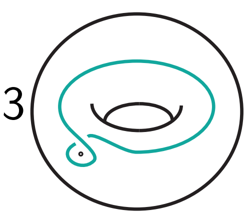
- **Example:** $[K] = \langle$  $\rangle$ and $T_3 = x^3 - 3x$,

$[K^{T_3}] =$ 

$=$  $- 3$  .

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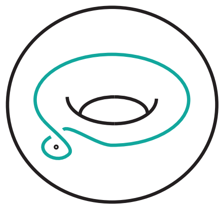
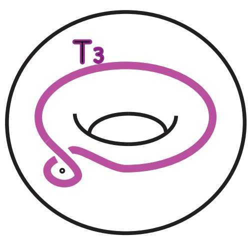
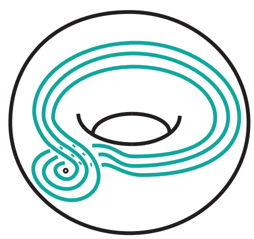
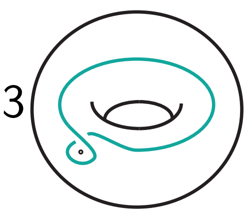
- Example: $[K] = \langle$  $\rangle$ and $T_3 = x^3 - 3x$,

$[K^{T_3}] =$ 

$=$  $- 3$  .

- Observe: For a framed link K with n crossings, the Kauffman bracket $[K]$ has 2^n resolutions, whereas $[K^{T_N}]$ has $\mathcal{O}(2^{n^2})$ resolutions.

Threading skeins by Chebyshev polynomials

Theorem (Bonahon-W.)

Let q be a N-root of 1, with N odd. Framed links threaded by T_N in $\mathcal{S}^q(\Sigma)$:

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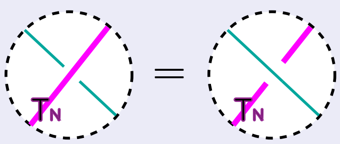
1. 

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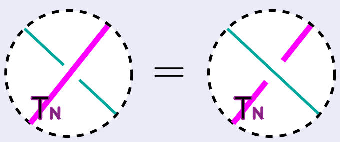
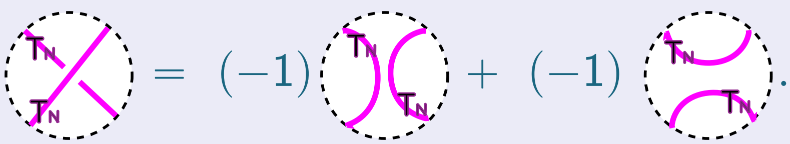
1. 

2. 

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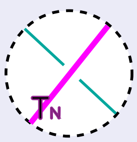
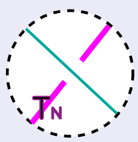
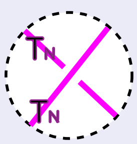
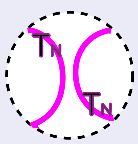
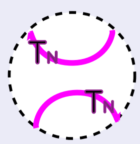
1. 

$=$ 

   $[K^{T_N}]$ are in the center.

2. 

$= (-1)$ 

$+ \ (-1)$ 

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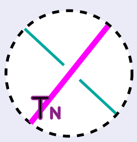
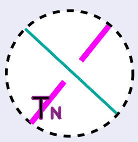
1. 

$=$ 

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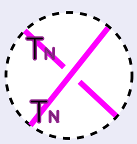
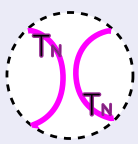
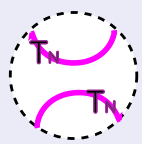
2. 

$= \ (-1)$ 

$+ \ (-1)$ 

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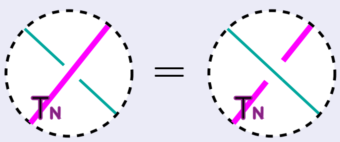
1. 

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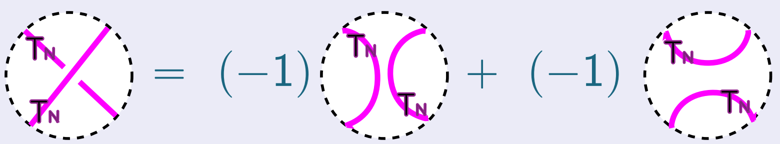
2. 

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Proof: Inject $\mathcal{S}^q(\Sigma) \hookrightarrow \mathcal{T}^q(\Sigma)$. There are many "miraculous cancellations" when checking identities in the quantum Teichmuller space $\mathcal{T}^q(\Sigma)$. (See also Le's skein-theoretic proof.)

Sketch Proof of Theorem

Theorem (Bonahon-W.)

Let q be a N-root of 1, with N odd.
Let Σ be a surface with finite topological type.

1. *Every irreducible representation of $\mathcal{S}^q(\Sigma)$ uniquely determines*
   - *puncture invariants* $p_i \in \mathbb{C}$ *for every puncture of* Σ
   - *classical shadow* $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$
2. *For compatible $p_i \in \mathbb{C}$ and $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$, there exists an irreducible representation of $\mathcal{S}^q(\Sigma)$ with those invariants.*

Sketch proof of Theorem, part 1

Suppose $\rho : \mathcal{S}^q(\Sigma) \to \mathrm{End}(\mathbb{C}^d)$ is irreducible.

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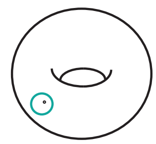
- The i-th puncture loop $[P_i]$ is central. 

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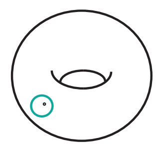
- The i-th puncture loop $[P_i]$ is central. 

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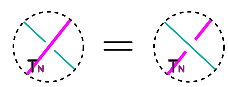
- A threaded link $[K^{T_N}] \in \mathcal{S}^q(\Sigma)$ is central. 
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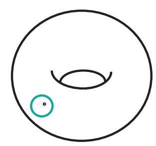
- The i-th puncture loop $[P_i]$ is central. 

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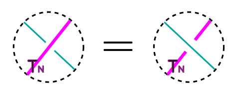
- A threaded link $[K^{T_N}] \in \mathcal{S}^q(\Sigma)$ is central. 

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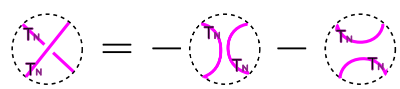
  $[K^{T_N}]$ satisfy the skein relation .

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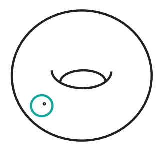
- The i-th puncture loop $[P_i]$ is central. 

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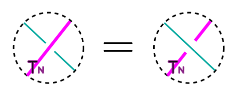
- A threaded link $[K^{T_N}] \in \mathcal{S}^q(\Sigma)$ is central. 
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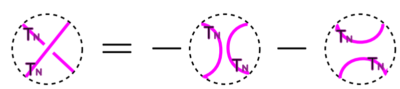
  $[K^{T_N}]$ satisfy the skein relation . So

$$\hat{\rho} : \mathcal{S}^1(\Sigma) \longrightarrow \mathbb{C}$$

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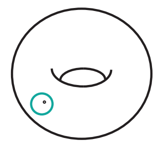
- The i-th puncture loop $[P_i]$ is central. 

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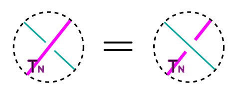
- A threaded link $[K^{T_N}] \in \mathcal{S}^q(\Sigma)$ is central. 

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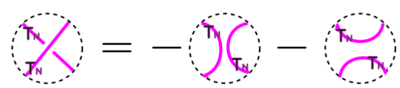
  $[K^{T_N}]$ satisfy the skein relation . So

  $$\hat{\rho} : \mathcal{S}^1(\Sigma) \longrightarrow \mathbb{C} \quad \longleftrightarrow \quad r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$$

  classical shadow

  from the isomorphism between $\mathcal{S}^q(\Sigma)$ and $\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)]$.

Sketch proof of Theorem, part 2

Suppose we are given compatible puncture invariants $p_i \in \mathbb{C}$ and classical shadow $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$.

Recall that for Σ with at least one puncture, the quantum Teichmuller space $\mathcal{T}^q(\Sigma)$ is the algebra generated by $Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}$ with $Z_i Z_j = q^{\pm 1} Z_j Z_i$ if the ith and jth edges share a triangle. Bonahon-Liu construct all representations of $\mathcal{T}^q(\Sigma)$, which are classified by compatible numbers $p_i \in \mathbb{C}$ and $r \in \mathcal{R}_{\mathrm{SL}_2\mathbb{C}}(\Sigma)$.

When Σ has at least one puncture, compose

$$\rho : \mathcal{S}^q(\Sigma) \hookrightarrow \mathcal{T}^q(\Sigma) \xrightarrow{p_i, r} \mathrm{End}(\mathbb{C}^d).$$

When Σ has no punctures, drill punctures and show there is an invariant subspace that is blind to punctures.

Further remarks

- In 2016 preprint, Frohman-Kania-Bartoszynska describe another method of constructing representations when Σ has at least one puncture, which matches our construction for generic characters by July 2017 preprint of Frohman-Kania-Bartoszynska-Le.

- $\mathcal{S}^q(\Sigma) \hookrightarrow \mathcal{T}^q(\Sigma)$ is used to prove many of the algebraic properties mentioned earlier (e.g., no zero divisors, determination of center).

- How can we use it to more closely tie together the Jones and Witten quantum invariants with hyperbolic geometry?

Thanks!