The SYK model, AdS_2 and conformal symmetry

Juan Maldacena

NatiFest

September 2016
Institute for Advanced Study

Nati and I collaborated on 8 papers.

Five were on aspects of two dimensional
String theory and their relation to matrix models.

An important paper by Nati

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hep-th/9705057

Comments on Higher Derivative Operators in Some SUSY Field Theories

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We study the leading irrelevant operators along the flat directions of certain supersymmetric theories. In particular, we focus on finite $N = 2$ (including $N = 4$) supersymmetric field theories in four dimensions and show that these operators are completely determined by the symmetries of the problem. This shows that they are generated only at one loop and are not renormalized beyond this order. An instanton computation in similar three dimensional theories shows that these terms are renormalized. Hence, the four dimensional non-renormalization theorem of these terms is not valid in three dimensions.

1995-1997

Andy Strominger:

The D3 brane near horizon geometry has maximal supersymmetry

16 → 32

This is double the expected number.

with no derivatives. If one examines instanton effects in the theory, it might seems that these could generate eight fermion operators at zero momentum. An instanton in this theory possesses 16 fermion zero modes, before including the effects of the Higgs fields. Out of the 16 supercharges 8 annihilate the classical solution and the other 8 generate zero modes. Similarly, out of the 16 superconformal symmetries 8 annihilate the classical instanton configuration and 8 generate zero modes. If one proceeds as in instanton calcula-

The extra supersymmetries → conformal symmetry → isometries of the near horizon region

AdS/CFT

The SYK model, AdS_2 and conformal symmetry

Based on work with
Douglas Stanford and Zhenbin Yang

Talks by Kitaev

Models of holography

Large N quantum system	Anomalous dimension	**Gravity/string dual**
• Free boundary theories.	$\gamma_{S>2} = 0$	• Bulk theories with massless higher spin fields.
• O(N) interacting theories.	$\gamma_{S>2} \sim 1/N$	• Very slighly massive higher spins
• Sachdev Ye Kitaev Model	$\gamma_{S>2} \sim 1$	• O(1) masses for the higher spin fields.
• Maximally supersymmetric Yang Mills at very strong t'Hooft coupling, g^2 N >> 1	$\gamma_{S>2} \gg 1$	• Einstein gravity theory. Higher spin particles are very massive.

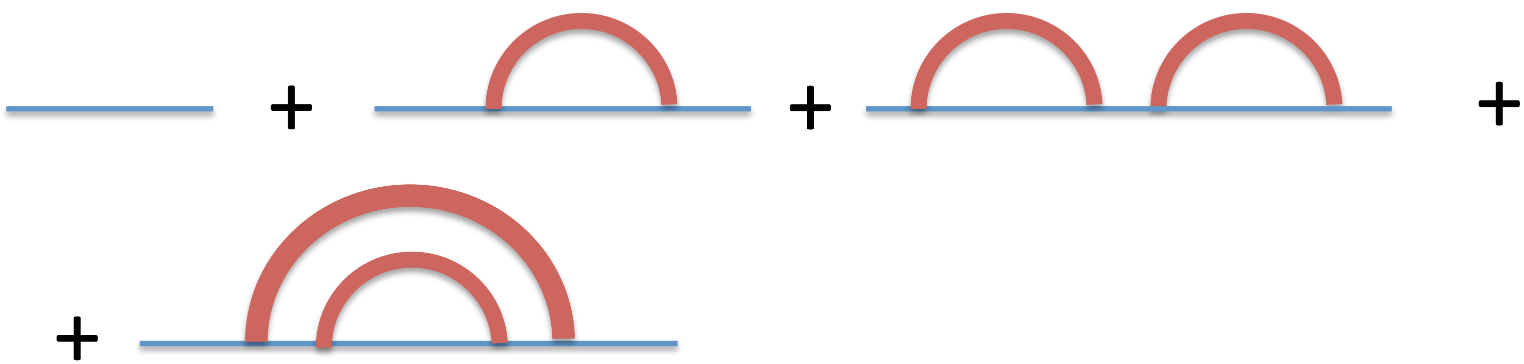
Harder

Easier

Solvable large N models

A simple solvable model

Rainbow diagrams.



Eg : 2d QCD, O(N) models,

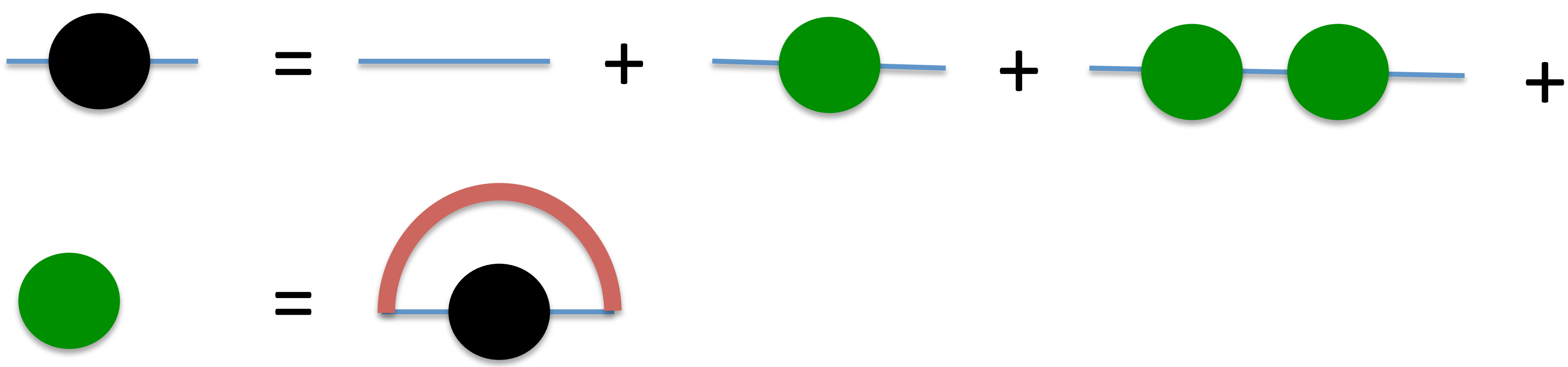
large N Chern Simons theories with fundamental matter in 2 +1 dimensions

Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin

Izuka Polchinski Okuda

Summing rainbow diagrams

Rainbow diagrams.



$$\frac{1}{\mathbf{G}(\omega)} = \frac{1}{G_0} - \Sigma(\omega)$$

$$\Sigma(t,t') = P(t,t')\mathbf{G}(t,t')$$

Special Case

$$P = J^2 = \text{constant}$$

Izuka Polchinski Okuda

Similar to what we get for the following model

N Majorana fermions $\{\psi_i, \psi_j\} = \delta_{ij}$

$$H = i \sum_{j,k} J_{jk} \psi_j \psi_k$$

$$\langle J_{ij}^2 \rangle = J^2/N$$

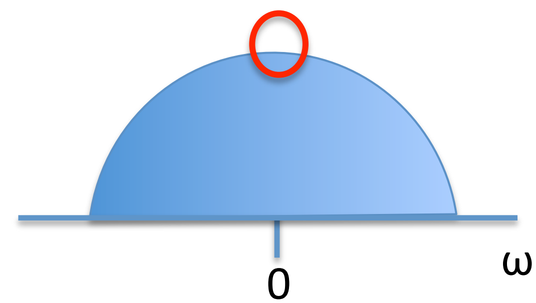
(no sum)

Random couplings, gaussian distribution.

To leading order → treat J as an additional field → same structure as before.

Solution of the special model

$$H = i \sum_{j,k} J_{jk} \psi_j \psi_k$$



Diagonalize J → semicircle distribution of energies.

Low energies → constant distributions → like a massless fermion on a circle of size N

Simple emergence of approximate scale invariance.

This model is too simple → no chaos, no black hole like-behavior.

The SYK model

N Majorana fermions $\{\psi_i, \psi_j\} = \delta_{ij}$

Sachdev Ye Kitaev
Georges, Parcollet

$$H = \sum_{i_1, \cdots, i_4} J_{i_1 i_2 i_3 i_4} \psi_{i_1} \psi_{i_2} \psi_{i_3} \psi_{i_4}$$

Random couplings, gaussian distribution. $\langle J^2_{i_1 i_2 i_3 i_4} \rangle = J^2 / N^3$

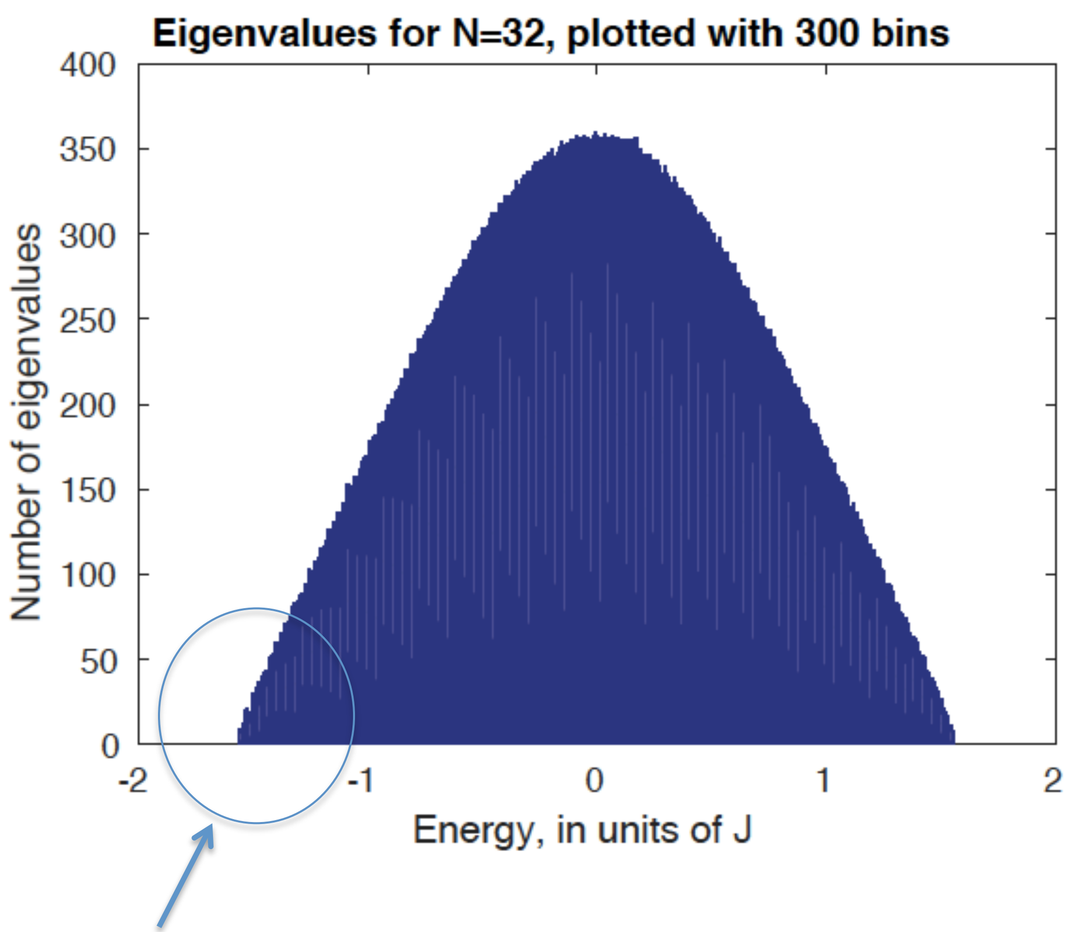
To leading order → treat J_{ijkl} as an additional field

J = dimensionful coupling. We will be interested in the strong coupling region

$$1 \ll \beta J, \ \tau J \ll N$$

Spectrum

D. Stanford



$$\dim_H = 2^{\frac{N}{2}}$$

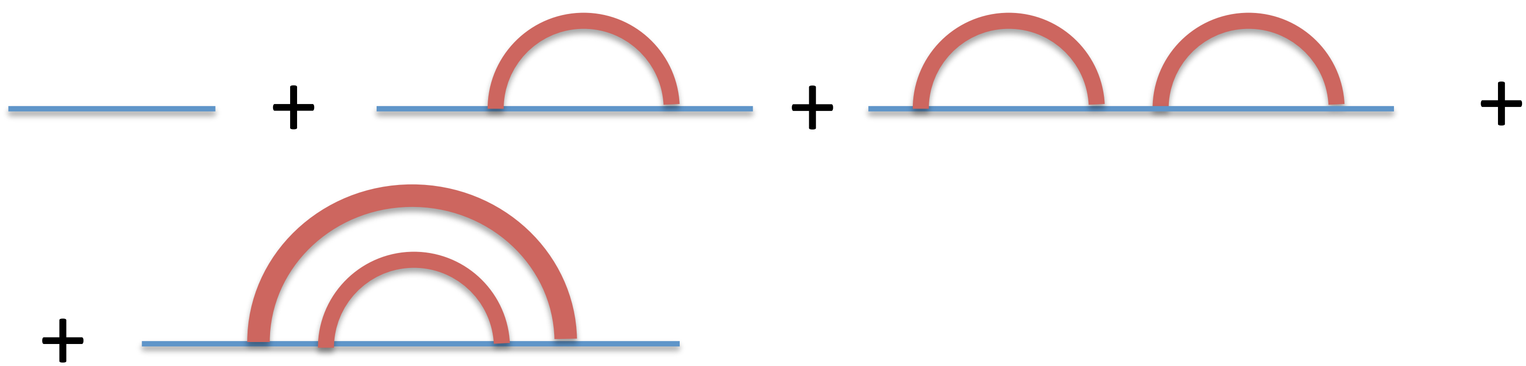
Number of random couplings $\propto N^4 \ll 2^N$

(specific, but random J's)

Exponentially large number of states contributes to the low energy region we consider

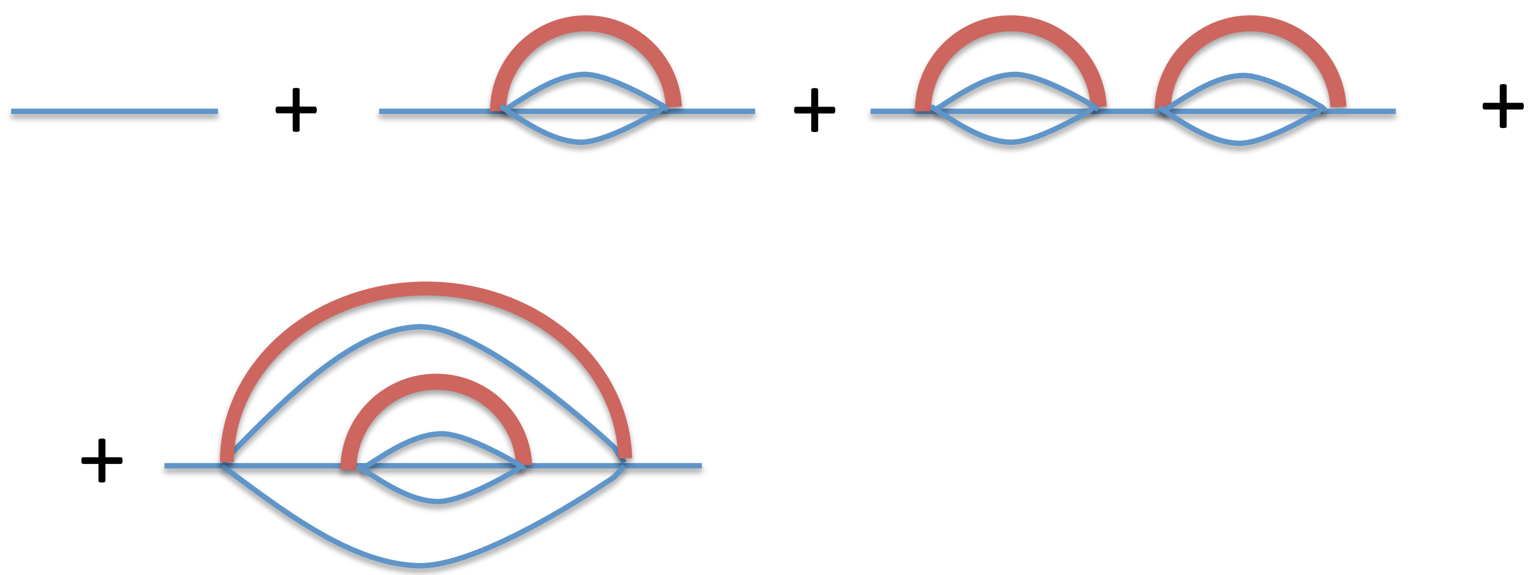
Large N limit

Before we had rainbows



Large N limit

Now:



$$\frac{1}{\mathbf{G}(\omega)} = \frac{1}{G_0} - \Sigma(\omega)$$

$$\Sigma(t,t') = J^2 \mathbf{G}(t,t')^3 \leftarrow$$

Generalization:

$$H = i^{q/2} \sum_{i_1, i_2, \cdots, i_q} J_{i_1, i_2, \cdots, i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$$

$$\frac{1}{\mathbf{G}(\omega)} = \frac{1}{G_0} - \Sigma(\omega)$$

$$\Sigma(t,t') = J^2 \mathbf{G}(t,t')^{q-1}$$

q=2 → case we had before.

q=4 → SYK

q = Infinity → analytically solvable equations.

In the IR → Conformal symmetry

Make a scale invariant ansatz

$$\frac{1}{\mathbf{G}(\omega)} = \cancel{\frac{1}{G_0}} - \Sigma(\omega)$$

$$\Sigma(t,t') = J^2 \mathbf{G}(t,t')^3$$

$$G_c(\tau,\tau') \propto \frac{1}{(\tau-\tau')^{2\Delta}}$$

is a solution if $\Delta = \frac{1}{q}$

If G is a solution, and we are given an arbitrary function f(τ), we can generate another solution:

$$G_c \longrightarrow G_{c,f}(\tau,\tau') = [f'(\tau)f'(\tau')]^{\Delta} G_c(f(\tau), f(\tau'))$$

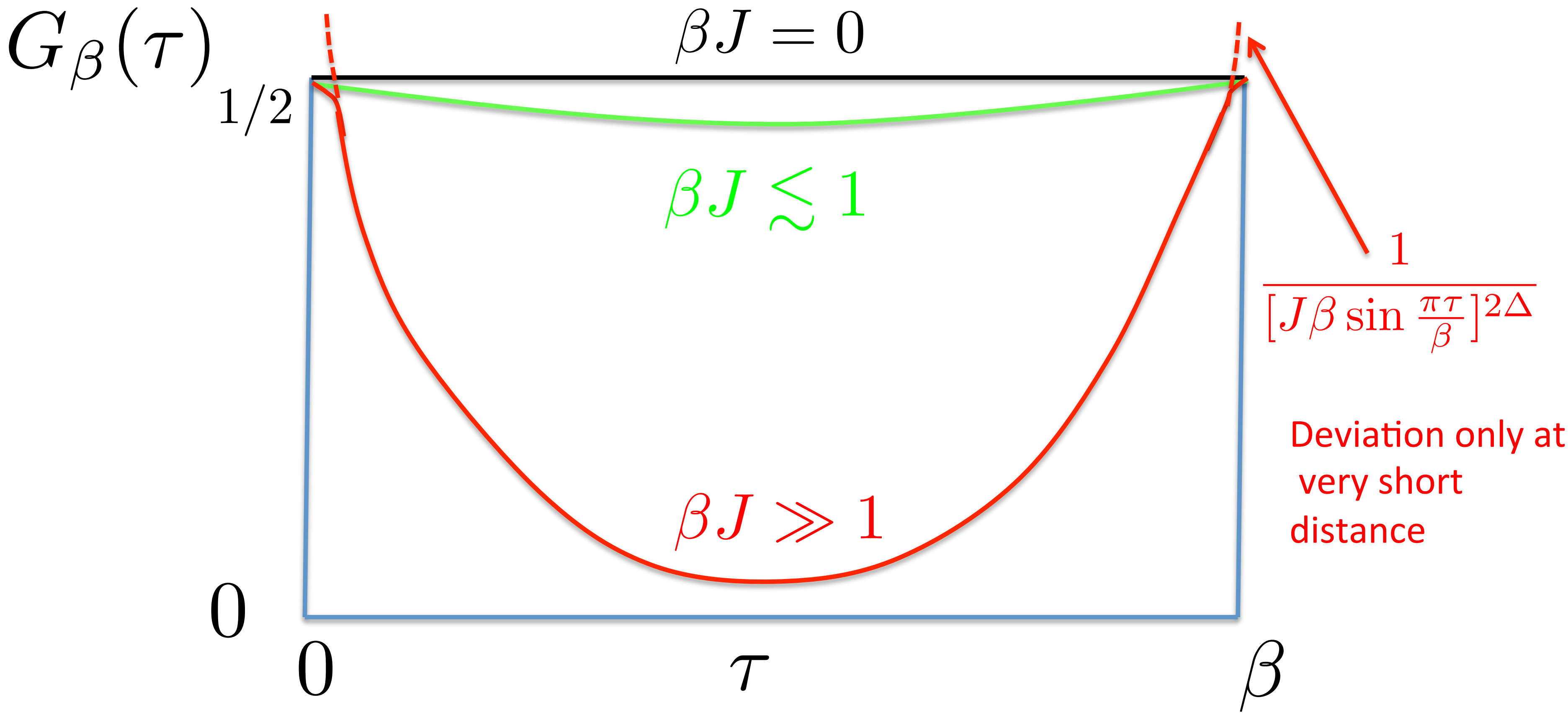
Example: Go from zero the temperature to a finite temperature solution

$$G(\tau,\tau') \propto \frac{1}{(\tau-\tau')^{2\Delta}}$$

$$f(\tau) = \tan\frac{\pi\tau}{\beta}$$

$$G_f = \left[\frac{\pi}{\beta\sin\frac{\pi\tau}{\beta}}\right]^{2\Delta}$$

General form of the propagator



Large N effective action

Integrate out the fermions and the couplings to obtain an effective action for the singlets, the fermion bilinears.

$$S = \frac{N}{2}\left[\log\det(\partial_t - \Sigma) - \int d\tau d\tau' \Sigma(\tau,\tau')G(\tau,\tau') + \frac{J^2}{4}G(\tau,\tau')^4\right]$$

Equations of motion from this action give the same as the Schwinger Dyson equations above.

Inserting the conformal answer here we get the extremal entropy.

It is non-local. The bilocal terms come from the integral over the couplings. $o(1/N^q)$

This effective action is correct to leading orders, where we can ignore the replicas,

Similar actions were obtained for usual O(N) style models.

Analog of the ``bulk action" : two dimensions, but det terms are not local. It is also the wrong space...

Zero modes of the action

Recall the conformal symmetry in the IR

$$G(\tau,\tau') \propto \frac{1}{(\tau-\tau')^{2\Delta}}$$

$$G \longrightarrow G_f(\tau,\tau') = [f'(\tau)f'(\tau')]^{\Delta} G(f(\tau),f(\tau'))$$

All these solutions have the same action.

Goldstone bosons → no action for f → will give a divergence if we do the path integral over f.

Solution: remember that the symmetry is also slightly broken.

Nearly zero modes of the action

$$G(\tau,\tau') \propto \frac{1}{(\tau-\tau')^{2\Delta}}$$

$$G \longrightarrow G_f(\tau,\tau') = [f'(\tau)f'(\tau')]^{\Delta} G(f(\tau), f(\tau'))$$

Deviations from the conformal solution were happening at short distances → expect that the effective action for f is local in time. We go from bilocal → local.

SL(2) invariance. $$f \to \frac{af+b}{cf+d}$$

Simplest action is ....

Schwarzian action

$$S = -\frac{N\alpha_s}{J}\int dt\,\mathrm{Sch}(f,t)\ , \qquad \mathrm{Sch}(f,t) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\frac{f''^2}{f'^2}$$

Numerical coefficient whose determination requires knowing the first deviation of the propagator from the IR conformal solution.
Can be computed numerically.

ghosts ?

Thinking of SL(2) as a gauge symmetry → removes ghosts of the higher derivative action.
goldstones = coset = Reparametrizations/SL(2)

Low energy

$$\int \mathcal{D}G\mathcal{D}\Sigma e^{-S[G,\Sigma]} \to \int \mathcal{D}f e^{-S_f} \int \mathcal{D}'G\mathcal{D}'\Sigma e^{-S_{\rm conf}[G,\Sigma]}$$

Measure fixed by SL(2) symmetry

Bagrets, Altland, Kamenev

Example: Four point function

disconnected

$$\langle \psi_i(\tau_1)\psi_i(\tau_2)\psi_j(\tau_3)\psi_j(\tau_4)\rangle \propto$$

$$\int \mathcal{D}G\mathcal{D}\Sigma G(\tau_1,\tau_2)G(\tau_3,\tau_4)e^{-S[G,\Sigma]} = G_c(\tau_1,\tau_2)G_c(\tau_3,\tau_4) + \frac{1}{N}\frac{1}{S_2}$$

Inverse of the quadratic action. Since the leading conformal answer has zero modes, this is enhanced. The enhanced terms are given by the Schwarzian action

Enhancement factor

$$\langle 4pt\rangle \propto \int \mathcal{D}f G_{c,f}(\tau_1,\tau_2)G_{c,f}(\tau_3,\tau_4)e^{-S_f} = \frac{\beta J}{N} F\left(\frac{\tau_1}{\beta},\frac{\tau_2}{\beta},\frac{\tau_3}{\beta},\frac{\tau_4}{\beta}\right)$$

$$G_{c,f}(\tau,\tau') = [f'(\tau)f'(\tau')]^{\Delta}[f(\tau)-f(\tau')]^{-2\Delta}$$

Four point function

$$\frac{\langle \psi_i(\tau_1)\psi_i(\tau_2)\psi_j(\tau_3)\psi_j(\tau_4)\rangle}{\langle \psi_i(\tau_1)\psi_i(\tau_2)\rangle\langle \psi_j(\tau_3)\psi_j(\tau_4)\rangle} = 1 + \frac{\beta J}{N} F\left(\frac{\tau_i}{\beta}\right)$$

We can use this to compute lorentzian four point functions by analytic continuation.

Different analytic continuations → different orders in Lorentzian signature.

Of particular interest is to compute the out of time order correlator that is responsible for the growth of commutators.

Shenker, Stanford, Kitaev

$$\frac{\langle \psi_i(0)\psi_j(\tau)\psi_i(0)\psi_j(\tau)\rangle}{\langle \psi_i(0)\psi_i(0)\rangle\langle \psi_j(\tau)\psi_j(\tau)\rangle} \propto 1 + i\frac{\beta J}{N} e^{\frac{2\pi\tau}{\beta}}$$

Exponential growth
Saturating chaos bound

JM, Shenker, Stanford

Full four point function

Can be computed by summing some ladder diagrams and using the conformal symmetry, after removing the Schwarzian contribution.

$$\langle 4pt \rangle \propto \frac{1}{N}\left[\beta J F\left(\frac{\tau_i}{\beta}\right) + \tilde{F}\left(\frac{\tau_i}{\beta}\right) + \frac{1}{(\sin\frac{\pi\tau_{12}}{\beta}\sin\frac{\pi\tau_{34}}{\beta})^{2\Delta}} H(\chi)\right]$$

Conformal invariant part → contains information about the operator spectrum.

Anomalous dimensions of higher spin fields are of order one.

$$\psi_i \partial^{1+2m}\psi_i \quad \rightarrow \quad h_m = 2\Delta + 1 + 2m + \gamma_m$$

$$1 = -(q-1)\frac{\Gamma(\frac{3}{2}-\frac{1}{q})\Gamma(1-\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})\Gamma(\frac{1}{q})}\frac{\Gamma(\frac{1}{q}+\frac{h}{2})}{\Gamma(\frac{3}{2}-\frac{1}{q}-\frac{h}{2})}\frac{\Gamma(\frac{1}{2}+\frac{1}{q}-\frac{h}{2})}{\Gamma(1-\frac{1}{q}+\frac{h}{2})}$$

Comparison with previous conformal quantum mechanics

$$\int dt(\dot{X}^2 + g/X^2)$$

De Alfaro, Fubini, Furlan

Michelson, Strominger, ….

Exact SL(2) symmetry acting on the dynamical variables.

No SL(2) invariant ground state.

Under a reparametrization the action changes as

$$\Delta S = \int d\tau X^2 \mathrm{Sch}(\mathrm{f}, \tau)$$

Different pattern of symmetry realization.

Is OK to describe brane probes in AdS_2, but does not seem to capture gravitational features properly.

Reparametrization symmetry

- SYK and AdS_2 both have an emergent, spontaneously broken and explicitly broken reparametrization symmetry.
- The spontaneous breaking, and the explicit breaking, both preserves an SL(2) gauge-like symmetry.

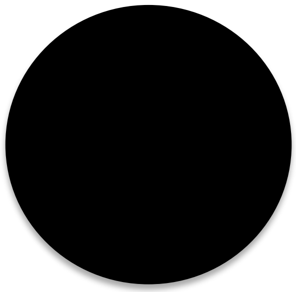
Questions

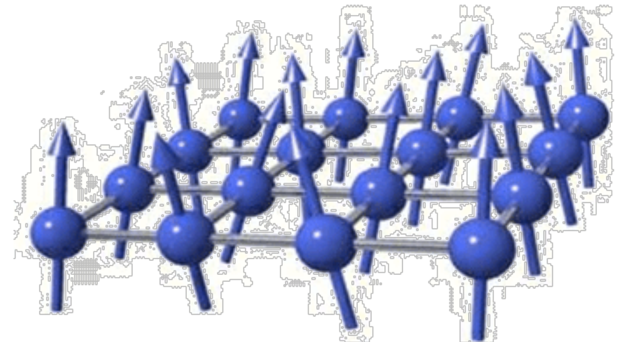
- The discussion was mostly through the Euclidean path integral.
- How should we think about this approximate symmetry in a Hilbert space context ?
- Is there a Virasoro algebra ?
- Is there any “central charge” to be computed ?

Reparametrizations in CFT_2

- In a CFT_2 we also have holomorphic reparametrizations $z \to f(z), \; \bar{z} \to \bar{f}(\bar{z})$
- Spontaneously broken by the vacuum to just $SL(2)^2$
- Goldstones: modes created by the stress tensor operator. They have a non-zero action consistent with conformal symmetry.
- Symmetry is not explicitly broken
- Symmetry algebra is deformed by the central charge (viewed as an operator with an expectation value = c) → Virasoro algebra.
- Only in the case that all other Δ>>1 it dominates.

Near extermal black holes





(Not a field theory.)

Extremal black hole $M \geq Q$ $M^2 \geq J$

Low energies, near horizon

AdS_2 region

Conformal quantum mechanics?

Conformal symmetry in quantum mechanics in a finite Hilbert space

- No go:

A. Strominger…

Density of states, scale invariance:

$$\rho(E) \propto \frac{1}{E} \,, \quad \text{or} \quad \delta(E) \quad ?$$

Either divergent in IR or no dynamics.

Gravity in two dimensions

- No go:

Naïve two dimensional gravity :

$$\int \sqrt{g}(R-2\Lambda)+S_M$$

Einstein term topological → no contribution to equations of motion.
Equations of motion → set stress tensor to zero.

No dynamics !

OK for extremal entropy. See Murthy's talk

Nearly AdS_2

Keep the leading effects that perturb away from AdS_2

Teitelboim Jackiw
Almheiri Polchinski

$$\int d^2x \sqrt{g}\phi(R+2) + \phi_0 \int d^2x \sqrt{g} R$$

Ground state entropy

Comes from the area of the additional dimensions, if we are getting this from 4 d gravity for a near extremal black hole.

$$\int \sqrt{g}\phi(R+2)$$

Equation of motion for ϕ → metric is AdS_2

Equation of motion for the metric → phi is almost completely fixed

$$ds^2 = d\rho^2 + \sinh^2\rho d\tau^2$$

$$\phi = \phi_h \cosh\rho$$

Value at the horizon

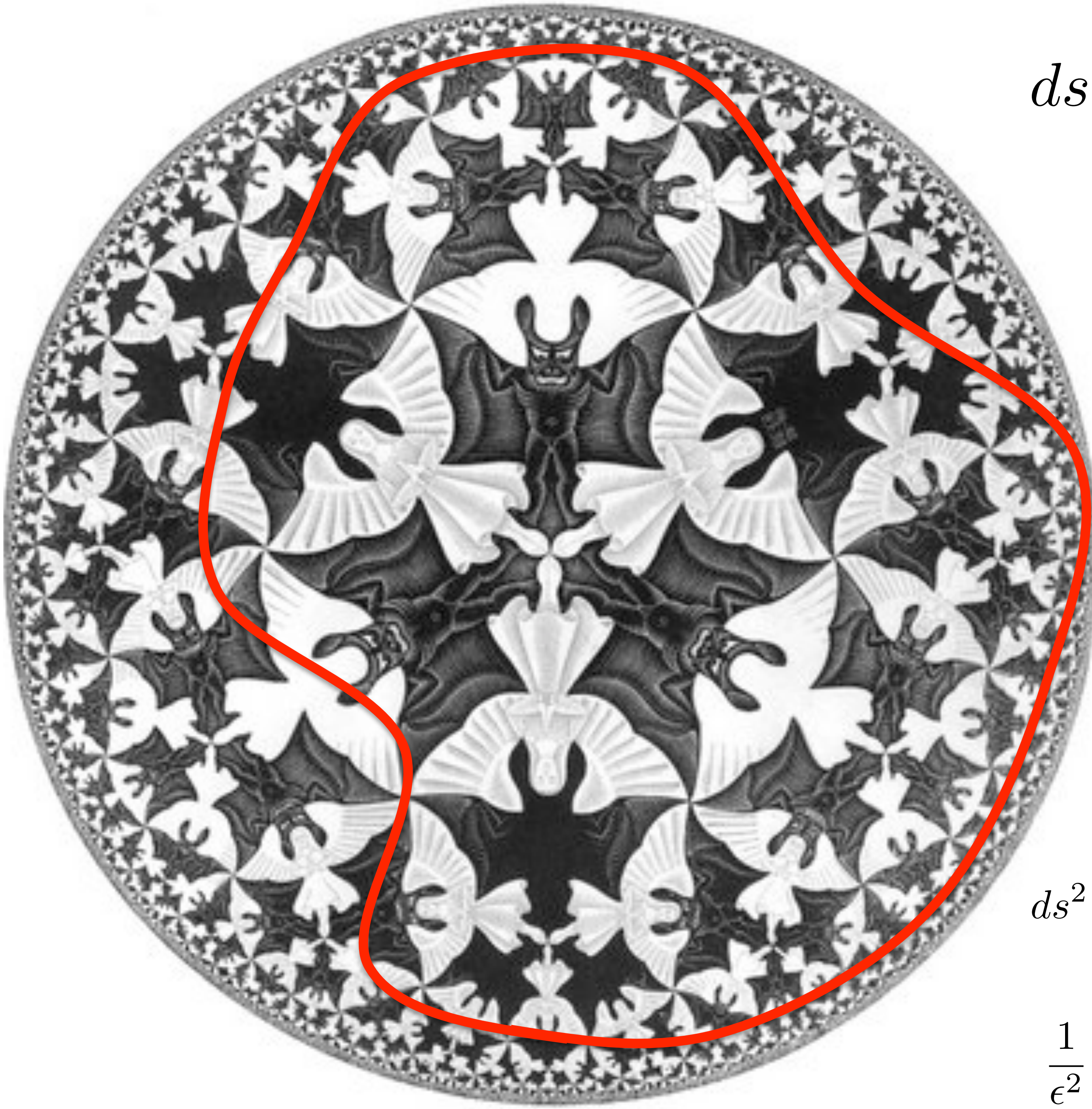
Position of the horizon.

In the full theory: when ϕ is sufficiently large → change to a new UV theory

Asymptotic boundary conditions:

$$ds^2|_{Bdy} = \frac{1}{\epsilon^2} du^2 \quad \longleftarrow \quad \text{Fixed proper length}$$

$$\phi|_{Bdy} = \frac{1}{\epsilon} \phi_r(u)$$



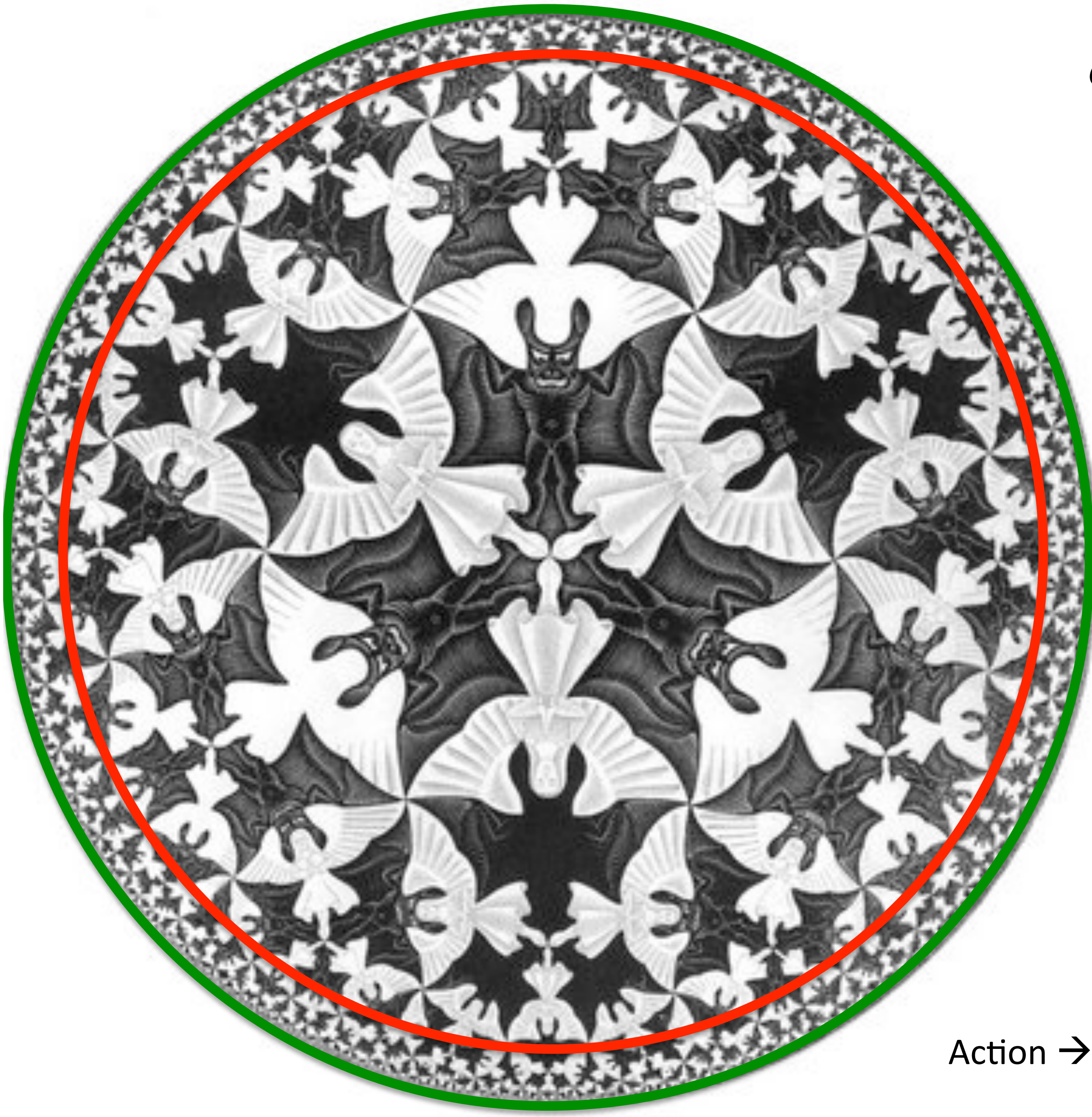
$$ds^2|_{Bdy} = \frac{1}{\epsilon^2} du^2$$

Infinite number of solutions.

$$ds^2 = d\rho^2 + \sinh^2 \rho d\tau^2$$

$$\rho(\tau)$$

$$\frac{1}{\epsilon^2} = (\rho'^2 + \sinh^2 \rho) \left(\frac{d\tau}{du}\right)^2$$



$$ds^2|_{Bdy} = \frac{1}{\epsilon^2} du^2$$

$$\phi|_{Bdy} = \frac{1}{\epsilon}\phi_r(u)$$

One one solution

Action → related to Schwarzian

$$S = \int d^2x\sqrt{g}\phi(R+2) - 2\int \frac{\phi_r(u)}{\epsilon^2} duK \to$$

$$S = \frac{1}{\epsilon^2} - \int du\phi_r(u)Sch(t,u)$$

$$t(u)$$

t = Usual AdS_2 time coordinate
u = Boundary system (quantum mechanical) time coordinate

Some qualitative relations

$$G_c(t_1, t_2) \propto \frac{1}{|t_1 - t_2|^{2\Delta}}$$

Background AdS_2 metric.

Both SL(2) invariant

Non-zero mode perturbations of G

Fields propagating on AdS_2

Nearly zero modes → t = f(u) , u is physical time , t = time set by the correlators, internal time.

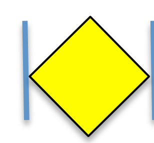
Gravitational interactions , via dilaton gravity → reduce to the same Schwarzian action.
t = AdS_2 coordinate time,
u = boundary proper time.

$$S[G, \Sigma]$$

$$S = S_{\rm dil.grav.} + S_{\rm matter}$$

$$G_c(t_L, t_R)$$

Wormhole or WdW patch of AdS_2



Properties fixed by the Schwarzian

- Common to $NAdS_2$ and in $NCFT_1$ (SYK, for example).
- Temperature dependence of the free energy $$S \propto \frac{N}{\beta J}$$
- Part of the four point function that comes from the explicit conformal symmetry breaking. This part leads to a chaos-like behavior with maximal growth in the commutator.

$$\text{growth of commutators} \sim \frac{1}{N}(\beta J) e^{2\pi t/\beta}$$

Kitaev

- Wormhole becomes traversable as we add a double trace interaction linking the two sides,

$$\int dt g(t) O_L(t) O_R(t)$$

Gao, Jafferis, Wall

All these properties are completely determined by the symmetries of the problem.

Happy Birthday Nati !

Extra slides

Bulk coordinates ?

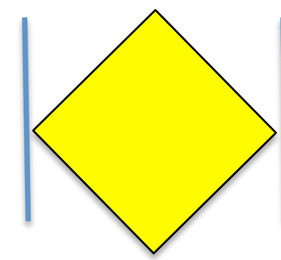
$$t = t_1 + t_2$$
$$z = t_1 - t_2$$

Conformal casimir acting on t_1 , t_2 → same as Wave operator on

$$ds^2 = \pm \frac{-dt^2 + dz^2}{z^2}$$

Finite temperature.

$$\tau = \frac{\tau_L + \tau_R}{2} \ , \qquad \sigma = \frac{\tau_L - \tau_R}{2} \qquad \longrightarrow \qquad ds^2 = \pm \frac{d\sigma^2 - d\tau^2}{\cosh^2 \tau}$$



Words of caution:

Does not work for Euclidean space.
It is not how it works in gravity. The two point functions are geodesics in space, not the space directly.

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hep-th/9705057

Comments on Higher Derivative Operators in Some SUSY Field Theories

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