Projective Dehn twist

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Symplectic manifolds

A symplectic manifold (M^{2n}, ω) is

- a smooth manifold M
- equipped with $\omega \in \Omega^2(M)$ such that $d\omega = 0$, ω^n nowhere vanishing

eg. $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)$, (T^*Q, ω_{can}), etc

Symplectomorphism

A symplectomorphism $\phi : (M, \omega_M) \to (N, \omega_N)$ is

- a diffeomorphism $\phi : M \to N$, such that
- $\phi^* \omega_N = \omega_M$

eg. when $(M, \omega_M) = (N, \omega_N)$ compact, time 1 flow along a vector field X on M such that $\mathcal{L}_X \omega_M = 0$

Dehn twist

Given a Lagrangian sphere $S \subset (M, \omega)$, one can perform Dehn twist $\tau_S : (M, \omega) \to (M, \omega)$ which is a symplectomorphism

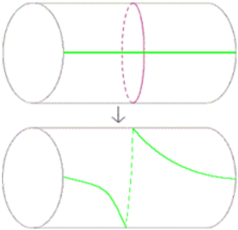

Figure : S is purple, L and $\tau_S(L)$ are green

Fukaya category

Fukaya category $\mathcal{F}uk(M,\omega)$ is an A_∞ category

- objects: Lagrangians submanifolds L (with additional structures/restrictions)
- morphism: $hom(L_0, L_1) = \oplus_{p \in L_0 \cap L_1} \mathbb{K} < p >$
- A_∞ operations $\mu_k : hom(L_{k-1}, L_k) \otimes \cdots \otimes hom(L_0, L_1) \to hom(L_0, L_k)$

Quasi-equivalence

Given an A_∞ category $\mathcal{A}$, one can take its cohomological category $H(\mathcal{A})$.
An A_∞ functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence if the induced functor on the cohomological category is an equivalence.

Seidel's exact triangle

Theorem (Seidel)

For any graded compact exact Lagrangian L, there is an exact triangle in $D^{\pi}\mathcal{F}uk(M,\omega)$

$$HF(S,L) \otimes S \to L \to \tau_S(L) \to HF(S,L) \otimes S[1]$$

The induced autoequivalence $T_S : D^{\pi}\mathcal{F}uk(M,\omega) \to D^{\pi}\mathcal{F}uk(M,\omega)$ by τ_S can be formulated purely algebraically.

Spherical twist

Let X be a smooth projective variety. An object $\mathcal{E}$ in $D^b(X)$ is spherical if

- $\mathcal{E} \otimes \omega_X \simeq \mathcal{E}$, and
- $Ext^*(\mathcal{E}, \mathcal{E}) = H^*(S^{dim(X)})$

A spherical object determines an autoequivalence $T_{\mathcal{E}}$ on $D^b(X)$.

Summary

Table : From symplectomorphism to autoequivalence

(M,ω)	$D^{\pi}\mathcal{F}uk(M,\omega)$	$D^b(X)$
S	S	$\mathcal{E}$
τ_S	T_S	$T_{\mathcal{E}}$

Questions

- Are there other symplectomorphisms supported near Lagrangian submanifolds?
- What can one say about the induced auto-equivalences?

Projective Dehn twist

A parallel story for projective space:

Table : From symplectomorphism to autoequivalence

(M, ω)	$D^\pi \mathcal{F}uk(M, \omega)$	$D^b(X)$
P	P	$\mathcal{P}$
τ_P	T_P	$T_{\mathcal{P}}$

Here, P is a Lagrangian (real/complex) projective space and $\mathcal{P}$ is a $\mathbb{P}$-object in $D^b(X)$.

Projective Dehn twist

The definition of $\mathbb{P}$-object and $\mathbb{P}$-twist is due to Huybrechts-Thomas and is motivated by the symplectomorphism τ_P. However, the relation between τ_P and T_P is still conjectural.

Conjecture (Huybrechts-Thomas)

The induced autoequivalence on $D^\pi \mathcal{F}uk(M, \omega)$ by τ_P is T_P.

Partial result

In monotone setting, using Mau-Wehrheim-Woodward functor and Biran-Cornea Lagrangian cobordism theory, we have

Theorem (M-Wu)

In $Tw\mathcal{F}uk(M,\omega)$, there is a natural quasi-isomorphism of objects $\tau_P(L) = Cone(Cone(hom(P,L) \otimes P[-2] \to hom(P,L) \otimes P) \to L)$ for every L

It looks similar to $T_P(L)$ but the morphisms in the theorem are not explicitly determined.

Question

Are spherical twists and $\mathbb{P}$-twists related?

A hybrid

A Lagrangian $S^2 = \mathbb{CP}^1$ in $D^\pi \mathcal{F}uk(M, \omega)$ is both spherical and projective (similarly $S^1 = \mathbb{RP}^1$)

Table : From symplectomorphism to autoequivalence

(M, ω)	$D^\pi \mathcal{F}uk(M, \omega)$	$D^b(X)$
$S = P$	$S = P$	$\mathcal{S} = \mathcal{P}$
$\tau_S^2 = \tau_P$	$T_S^2 = T_P$	$T_{\mathcal{S}}^2 = T_{\mathcal{P}}$

Another hybrid

A Lagrangian $P = \mathbb{RP}^{2n+1}$ is

- a $\mathbb{P}$-object when $char(\mathbb{K}) = 2$
- a spherical object when $char(\mathbb{K}) \neq 2$

Question

What is the induced autoequivalence of τ_P on $D^\pi \mathcal{F}uk(M, \omega)$ when $char(\mathbb{K}) \neq 2$?

Hints

When $char(\mathbb{K}) \neq 2$, the induced autoequivalence by τ_P on $D^\pi \mathcal{F}uk(M, \omega)$ is

- well-defined when P is (relatively) spin
- not a projective twist
- not a spherical twist
- not a square of a spherical twist

Work in progress

Theorem (M-Wu)

Let $P = \mathbb{RP}^{4n+3}$ be a monotone Lagrangian in a close monotone (M, ω). The induced autoequivalence by τ_P is a simultaneous spherical twist by two spherical objects when char$(\mathbb{K}) \neq 2$.

Goal:

- Explain to you autoequivalences obtained by S^n/Γ twists in various characteristic
- Observe new phenomena of autoequivalences of $D^b(X)$

THANK YOU

Thank you very much for your attention !