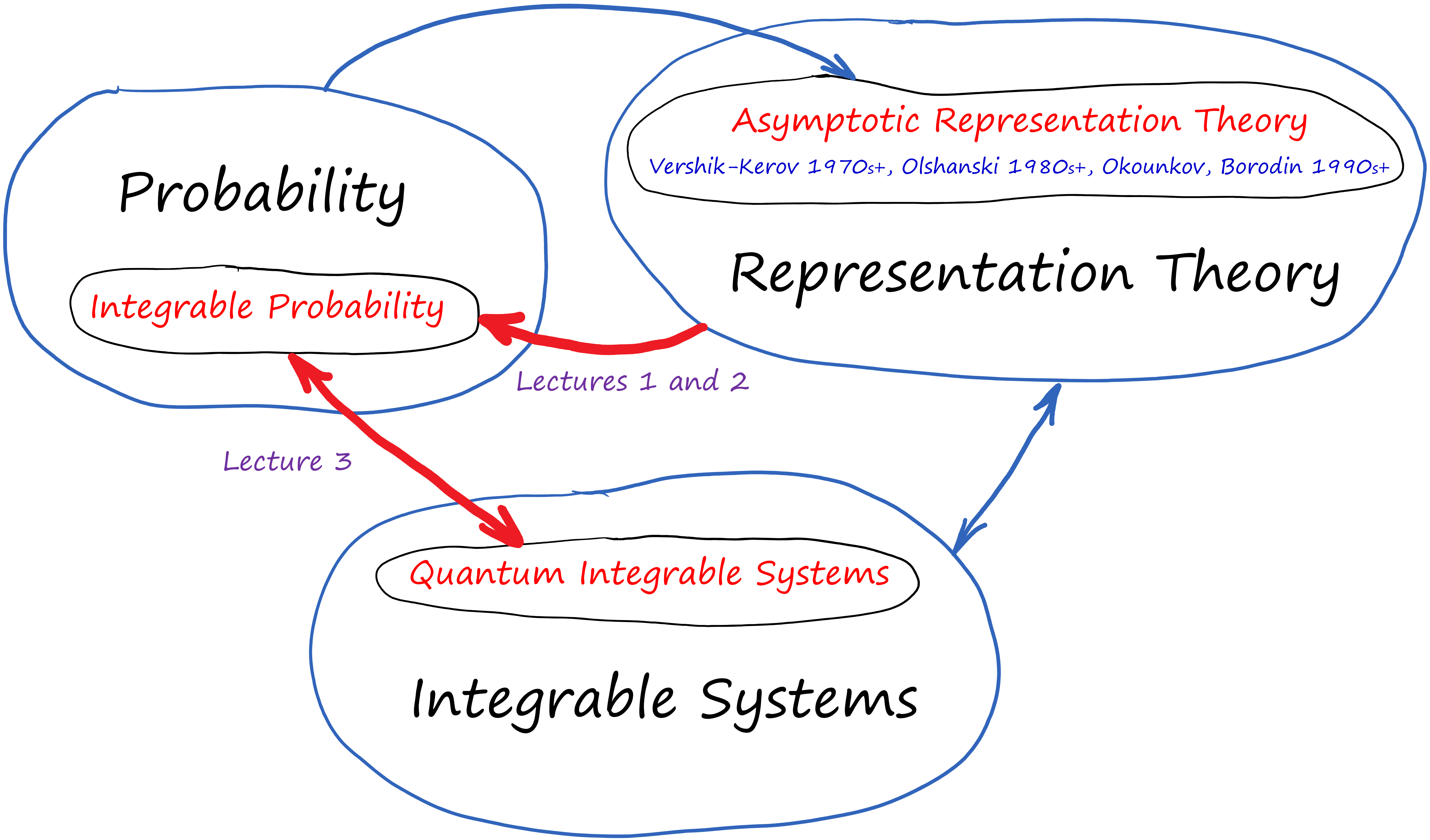
Macdonald processes

Alexei Borodin


Asymptotic Representation Theory
Vershik-Kerov 1970s+, Olshanski 1980s+, Okounkov, Borodin 1990s+
Probability
Integrable Probability
Representation Theory
Lectures 1 and 2
Lecture 3
Quantum Integrable Systems
Integrable Systems

Probabilistic objectives

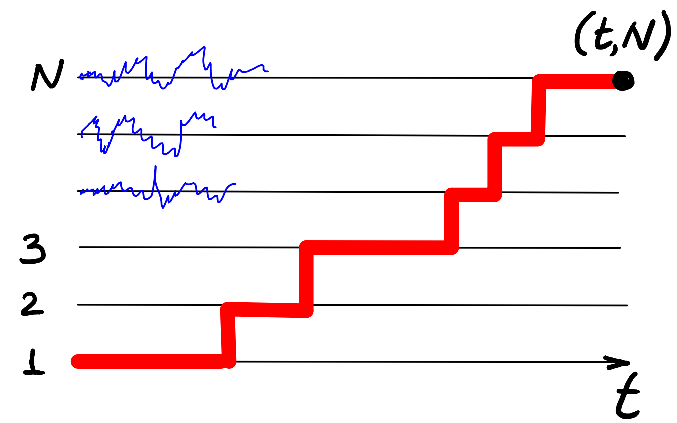
We wish to establish law of large numbers and fluctuations behaviour for a (growing) variety of integrable probabilistic models that have an additional algebraic structure, like

- Random matrix ensembles with rotational symmetry
- Exclusion processes in (1+1)d: TASEP, ASEP, PushASEP, q-versions, etc.
- Special directed random polymers in (1+1)d
- Special tiling (or dimer) models in 2d
- Random growth of discretized interfaces in (2+1)d

Universality principles suggest that same fluctuations hold in broad universality classes (Wigner matrices, KPZ, general dimers)

Example 1: Semi-discrete Brownian polymer

$$F_t^N = \log \int\limits_{0<s_1<\dots<s_{N-1}<t} e^{B_1(0,s_1)+B_2(s_1,s_2)+\dots+B_N(s_{N-1},t)}\, ds_1 \dots ds_{N-1}$$



$B_1, \dots, B_N$ are independent Brownian motions

$$B_k(\alpha,\beta) := B_k(\beta) - B_k(\alpha) = \int_\alpha^\beta \dot{B}_k(x)dx$$

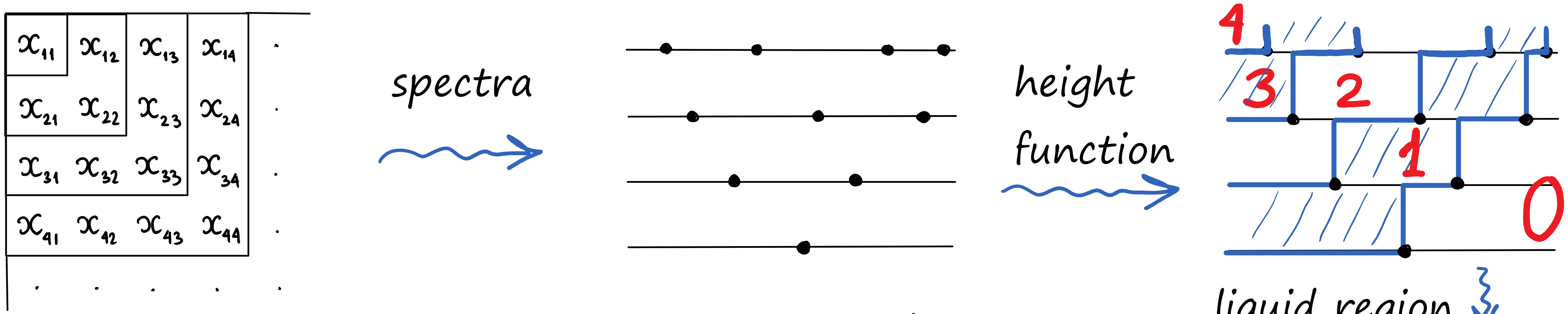
Theorem [B-Corwin '11, B-Corwin-Ferrari '12] For any $\varkappa > 0$

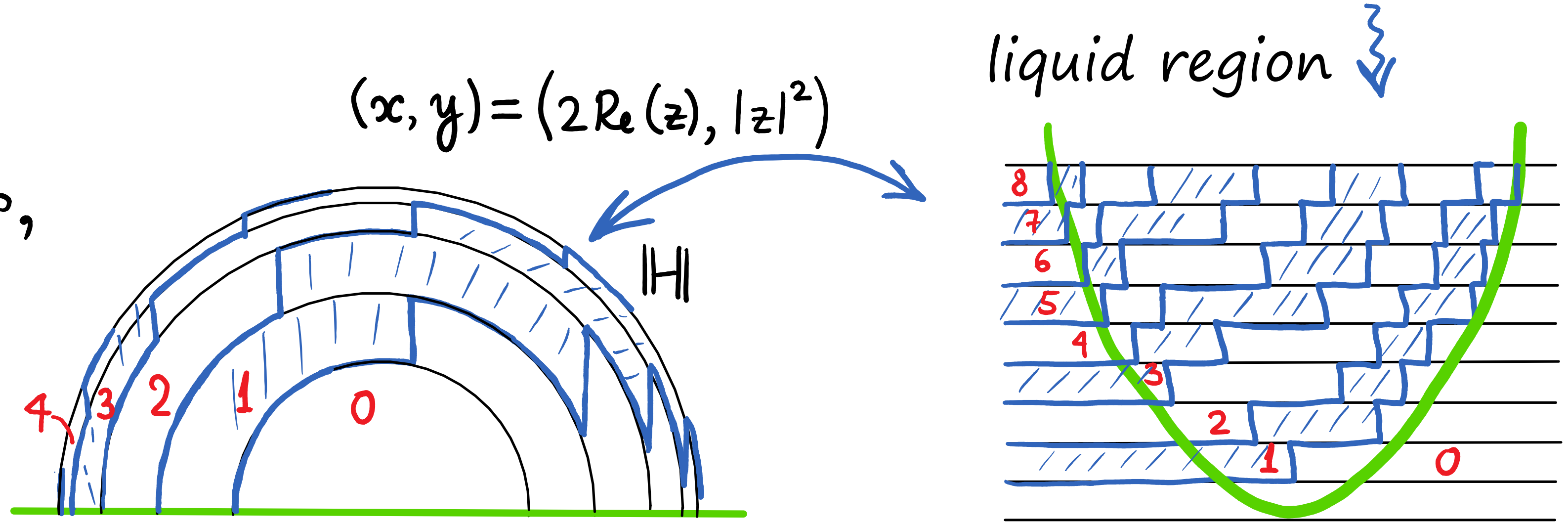
$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{F^N_{\varkappa N} - f_\varkappa \cdot N}{g_\varkappa \cdot N^{1/3}} \le r \right\} = F_{GUE}(r)$$

Tracy-Widom limit distribution for the largest eigenvalue of large Hermitian random matrices

- $f_\varkappa$ conjectured in [O'Connell-Yor '01], proved in [Moriarty-O'Connell '07]
- [Spohn '12] matched the result with (1+1)d KPZ scaling conjecture

Example 2: Corners of random matrices





__Theorem__ As $z \mapsto L^{-1}z$, $L \to \infty$,
Fluctuations $\Rightarrow$
Gaussian (massless)
Free Field on $\mathbb{H}$

- GUE: Implicit in [B-Ferrari, 2008], related to AKPZ in (2+1)d
- GUE/GOE type Wigner matrices : [B, 2010]
- General beta, classical weights : [B-Gorin, 2013]

Two characteristic properties

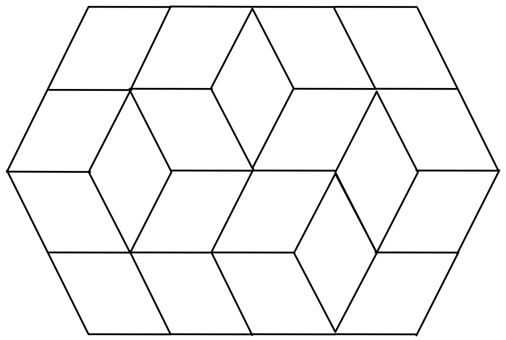
Integrable probabilistic models typically share two key features:

- There is a large family of observables whose averages are explicit and asymptotically tractable;
- There is a natural Markov evolution that acts nicely.

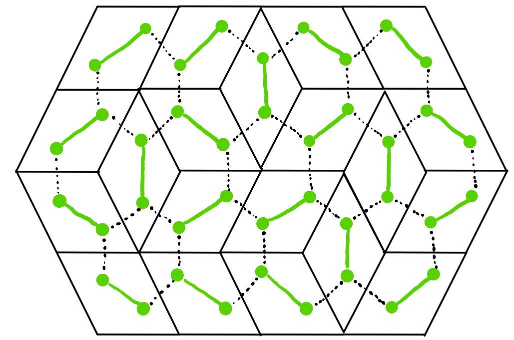
Representation theory is helpful in identifying both.

Let us illustrate on lozenge tilings.

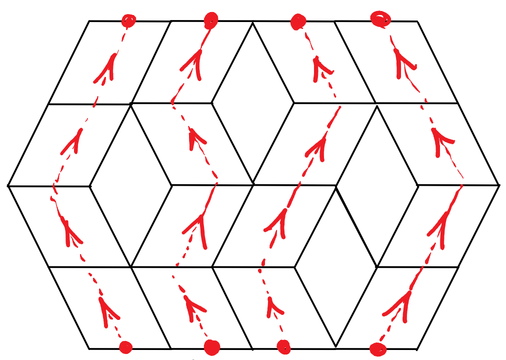
From probability to representation theory



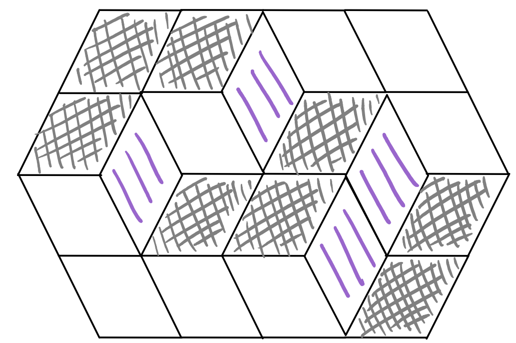
Lozenge tilings are...



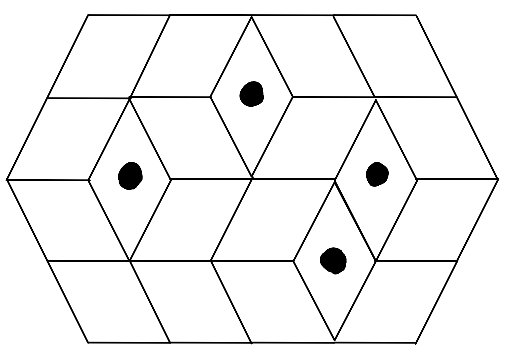
dimers on hexagonal lattice



nonintersecting Bernoulli paths



stepped surfaces



interlacing particle configurations

But they are also labels for Gelfand-Tsetlin bases of irreps of $U(N)$ or $GL(N,\mathbb{C})$.

Finite-dim representations of unitary groups (H. Weyl, 1925-26)

A *representation* of $U(N)$ is a group homomorphism $T: U(N) \to GL(V)$.

It is *irreducible* if V has no invariant subspaces.

Every (finite-dimensional) representation is a direct sum of irreps.

Fact: T is *uniquely determined* by the (diagonalizable) action of the abelian subgroup H of diagonal matrices.

$$V = \bigoplus_{i=1}^{\dim V} \mathbb{C} v_i, \qquad T\left(\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{bmatrix}\right) = \begin{bmatrix} t_1(z_1,\dots,z_N) & & \\ & \ddots & \\ & & t_{\dim V}(z_1,\dots,z_N) \end{bmatrix}$$

$$t_j : S^1 \times \dots \times S^1 \to \mathbb{C}^\times, \qquad t_j(z_1,\dots,z_N) = z_1^{k_1} \cdots z_N^{k_N}, \qquad \underset{\text{weight}}{(k_1,\dots,k_N)} \in \mathbb{Z}^N.$$

Finite-dim representations of unitary groups (H. Weyl, 1925–26)

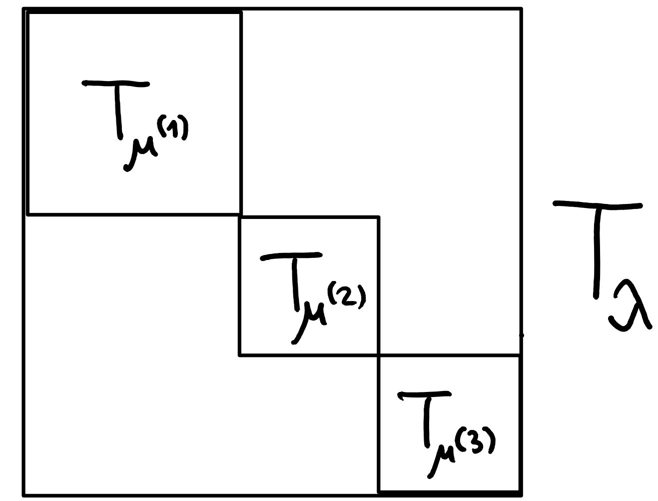
Theorem Irreducible representations are parametrized by their highest weights $\lambda = (\lambda_1 \geq \dots \geq \lambda_N) \in \mathbb{Z}^N$. The corresponding generating function of all weights has the form

$$\sum_{\text{weights of } T_\lambda} z_1^{k_1} \cdots z_N^{k_N} = \mathrm{Trace}\left(T_\lambda\left(\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{bmatrix}\right)\right) = \frac{\det\left[z_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\underbrace{\det\left[z_i^{N-j}\right]_{i,j=1}^N}_{\text{Vandermonde det.}}}.$$

These are the characters of the corresponding representations, also known as the Schur polynomials.

Branching and lozenges

Reducing the symmetry group from U(N) to U(N−1) may lead to a split of an irrep into a direct sum of those for the smaller group.

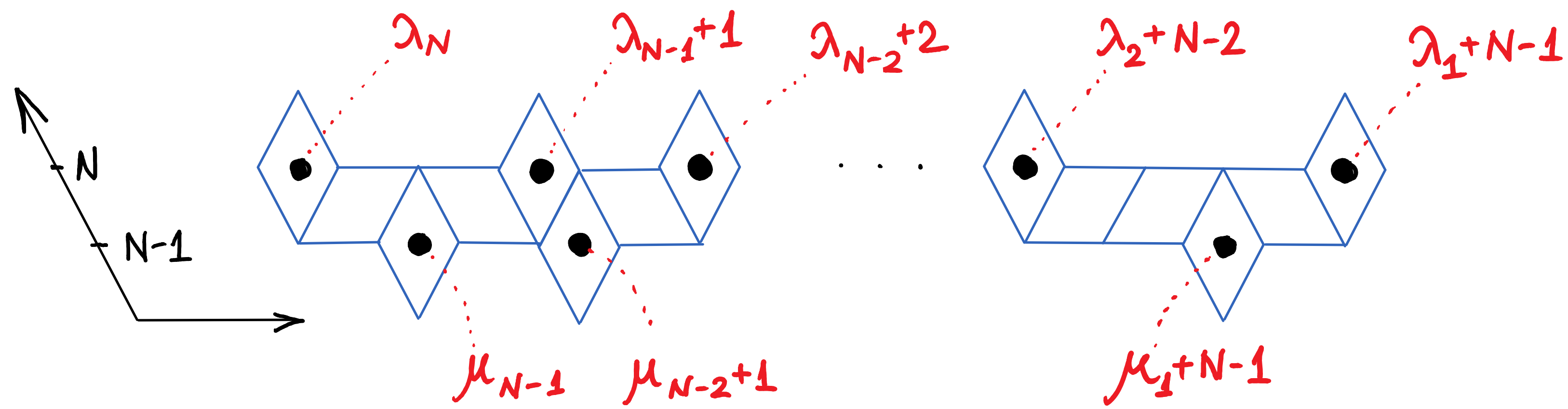


This is encoded by Schur polynomials:

$$S_\lambda(z_1,\dots,z_{N-1},1) = \sum_{\mu \prec \lambda} S_\mu(z_1,\dots,z_{N-1})$$

where μ interlaces λ : $\lambda_N \leq \mu_{N-1} \leq \lambda_{N-1} \leq \dots \leq \lambda_2 \leq \mu_2 \leq \lambda_1$,

or pictorially:

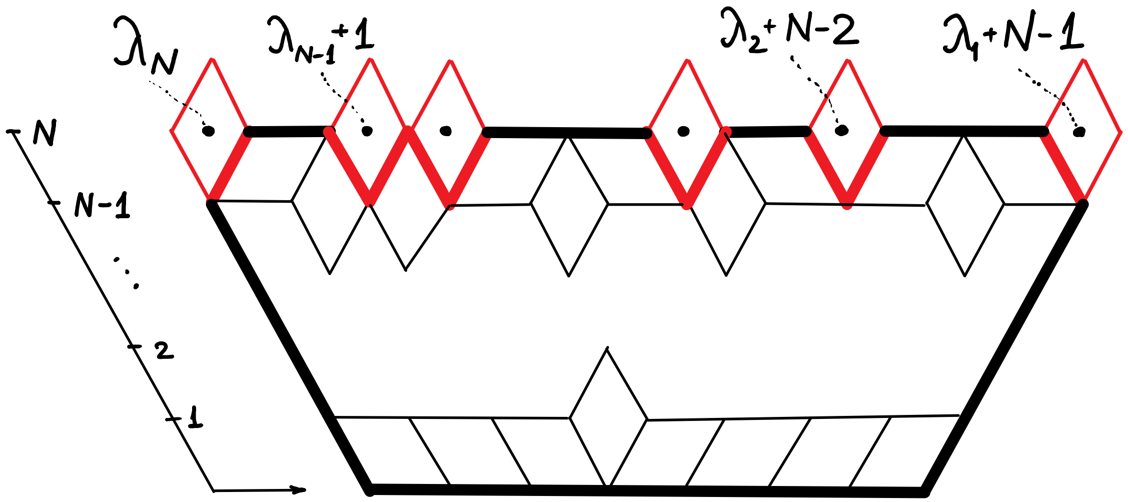


Gelfand-Tsetlin basis

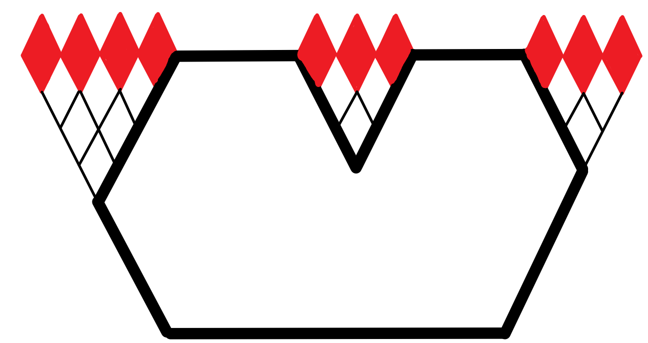
Reducing the symmetry all the way down the tower

$$U(N) \supset U(N-1) \supset \ldots \supset U(2) \supset U(1)$$

yields a basis in T_λ labelled by lozenge tilings of specific domains:



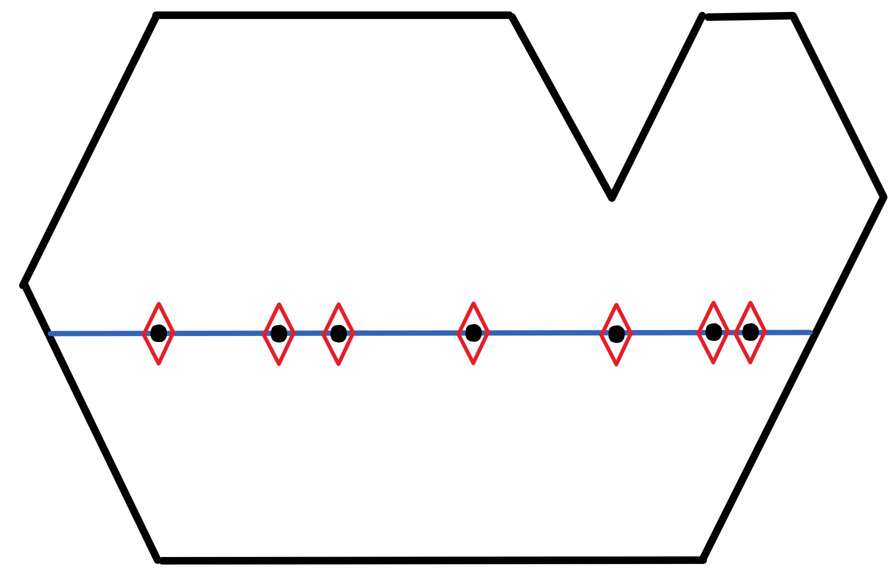
An example:



[Gelfand-Tsetlin, 1950] used this basis to explicitly write down the action of generators.

Back to probability

Consider the uniform measure on tilings.
How to describe its projection to a
horizontal section of the polygon?
Equivalently, how to decompose a known
irrep of $U(N)$ on irreps of $U(k) \subset U(N)$?



This is a problem of noncommutative harmonic analysis. In terms of characters (Schur polynomials):

$$\chi(z_1,\dots,z_k) = \sum_{\mu=(\mu_1\ge\dots\ge\mu_k)} \mathrm{Prob}\{\mu\} \frac{S_\mu(z_1,\dots,z_k)}{S_\mu(1,\dots,1)}, \qquad \chi(z_1,\dots,z_k) = \frac{S_\lambda(\overbrace{z_1,\dots,z_k,1,\dots,1}^{N})}{S_\lambda(1,\dots,1)}.$$

Classical harmonic analysis

The (abelian) group $\mathbb{R}$ acts on $L^2(\mathbb{R})$ by shifting the argument.

The irreps are all 1-dim of the form $p \mapsto$ multiplication by e^{-ipx}.

For

$$\chi(x) = \int_{-\infty}^{+\infty} e^{-ipx}\, m(dp)$$

there are (at least) two ways to extract information about m.

Inverse Fourier transform: $$\frac{m(dp)}{dp} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\chi(x)\,dx \quad \text{(hard)}$$

Differential operators: $$\int_{-\infty}^{\infty} p^n\, m(dp) = \left(i\frac{d}{dx}\right)^n \chi(x)\Big|_{x=0} \quad \text{(simple)}$$

The observables

If

$$\chi(z_1,\dots,z_k) = \sum_{\mu=(\mu_1\geq\dots\geq\mu_k)} \mathrm{Prob}\{\mu\} \frac{S_\mu(z_1,\dots,z_k)}{S_\mu(1,\dots,1)}$$

and $\mathcal{D} S_\mu = d_\mu S_\mu$, then $\mathcal{D}\chi\Big|_{z_1=\dots=z_k=1} = \sum_\mu d_\mu \,\mathrm{Prob}\{\mu\} = \mathbb{E}\, d_\mu$.

The *Casimir-Laplace operator* (generates circular Dyson BM)

$$C_2 = \frac{1}{\prod_{i<j}(z_i-z_j)} \circ \sum_{i=1}^{k}\left(z_i\frac{\partial}{\partial z_i}\right)^2 \circ \prod_{i<j}(z_i-z_j).$$

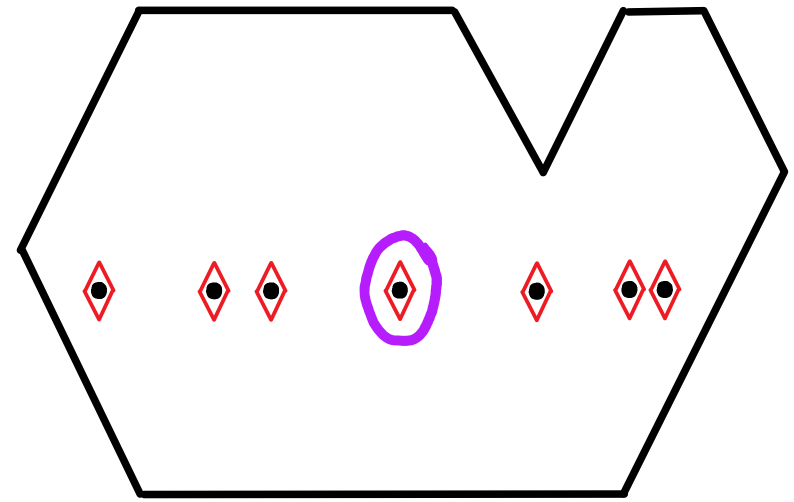
As $S_\mu(z) = \det\left[z_i^{\mu_j+k-j}\right] / \prod_{i<j}(z_i-z_j)$, $\quad C_2 S_\mu = \sum_{i=1}^{k}(\mu_i+k-i)^2 \cdot S_\mu$.

d_μ

A q-analog: Replace $\left(z\frac{\partial}{\partial z}\right)^2$ by $(T_q f)(z) = f(qz)$. Then $C^{(q)} S_\mu = \sum_{i=1}^{k} q^{\mu_i+k-i} \cdot S_\mu$.

Correlation functions

First correlation function:



$$\rho_1(m,k) = \mathrm{Prob}\{m \in \{\mu_j + k - j\}_{j=1}^{k}\} =$$

$$= \text{coeff. of } q^m \text{ in } \mathbb{E}\left(\sum q^{\mu_j+k-j}\right)$$

$$= \text{coeff. of } q^m \text{ in } C^{(q)}\chi\Big|_{z_1=\dots=z_k=1}.$$

Higher correlation functions require products $C^{(q_1)} \cdots C^{(q_n)}$.

If χ factorizes, $\chi(z_1,\dots z_k) = \varphi(z_1)\cdots\varphi(z_k)$,

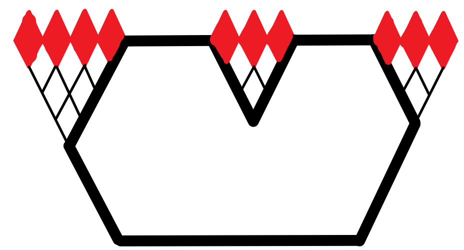
$$\text{coeff. of } q^m \text{ in } C^{(q)}\chi\Big|_{z_j\equiv 1} = \frac{1}{(2\pi i)^2}\oint_{\text{around } 0}\frac{dv}{v}\oint_{\text{around } 1} dw\, \frac{\varphi(v)\,(v-1)^k\, v^{-m}}{\varphi(w)\,(w-1)^k\, w^{-m}}\,\frac{1}{v-w}.$$

For the n-point correlation function the integral is 2n-fold.

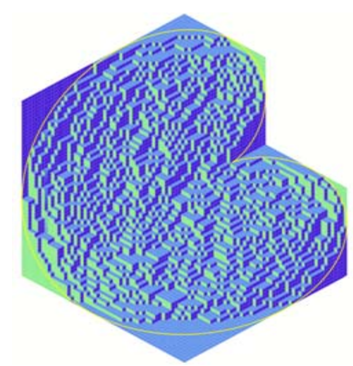
Asymptotics

For `infinitely tall polygons' (corresponding to characters of $U(\infty)$, example on next slide), χ indeed factorizes, and steepest descent yields limit shapes, bulk (discrete sine), edge (GUE, Airy, Pearcey), and global (free field) fluctuations [B-Kuan '07], [B-Ferrari '08].

For ordinary polygons in our class, the factorization is only approximate, yet same formulas can be used to prove similar results [Petrov '12], [Gorin-Panova '13].



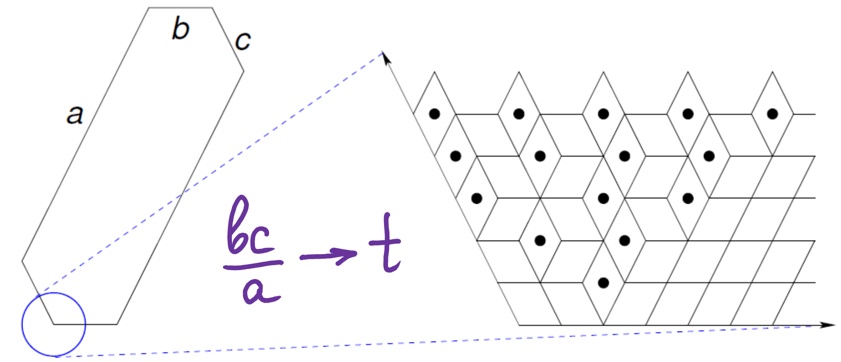
More general limit shapes were obtained by [Kenyon-Okounkov '05], who also conjectured the rest.



Markov evolution

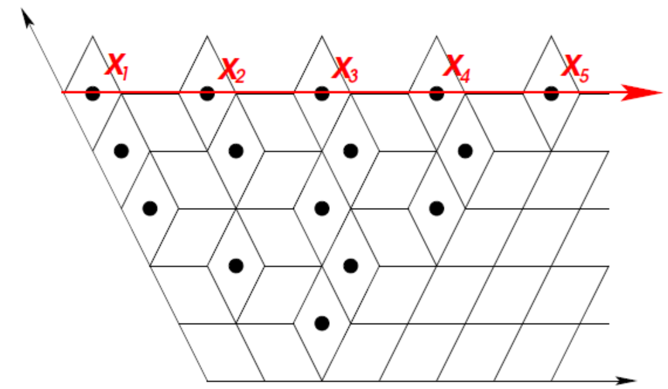
We focus on $\chi(z_1,\dots z_k)=\prod_{i=1}^{k} e^{t(z_i-1)}$, $t\geq 0$.

This corresponds to a limit of hexagons:



On a fixed horizontal slice, the coordinates of vertical lozenges are distributed as

$$\mathrm{Prob}\left\{(x_1,\dots,x_k)\in\mathbb{Z}^k_{\geq 0}\right\}=\mathrm{const}\cdot\prod_{i<j}(x_i-x_j)^2\prod_{i=1}^{k}\frac{t^{x_i}}{x_i!}.$$



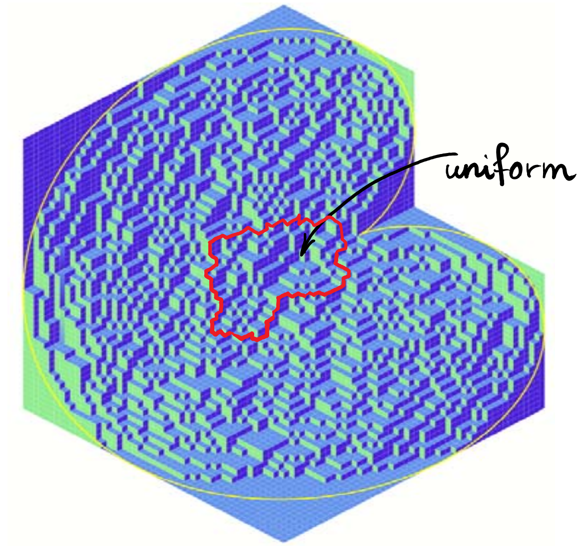
This is time t distribution of the Markov chain with generator

$$L^{(k)}_{\mathrm{Poisson}}=\left(\prod_{i<j}(x_i-x_j)\right)^{-1}\circ\sum_{i=1}^{k}\nabla_{x_i}\circ\left(\prod_{i<j}(x_i-x_j)\right),\qquad (\nabla f)(x)=f(x+1)-f(x),$$

which can also be viewed as k conditioned 1d Poisson processes.

The Gibbs property

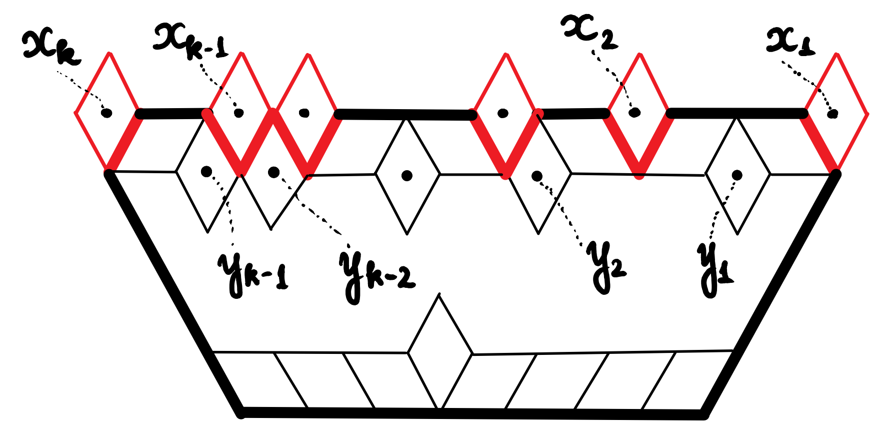
Uniformly distributed tilings obviously enjoy the Gibbs property: Given a boundary condition, the distribution in any subdomain is also uniform.



Apply to bottom k rows:

$$\mathrm{Prob}\{y \mid x\} = \frac{\#\text{ of height } (k-1) \text{ tilings with top row } y}{\#\text{ of height } k \text{ tilings with top row } x}$$

$$= (k-1)!\,\frac{\prod_{1\le i<j\le k-1}(y_i-y_j)}{\prod_{1\le i<j\le k}(x_i-x_j)} =: \Lambda^{k}_{k-1}(x \searrow y)$$



These stochastic links intertwine \`perpendicular' Markov chains along (k-1)st and k-th rows with generators $L^{(k-1)}_{\text{Poisson}}$ and $L^{(k)}_{\text{Poisson}}$.

Two-dimensional Markov evolution: Axiomatics

Inspired by two *ad hoc* constructions (RSK and [O'Connell '03+]; `stitching' of intertwined Markov chains [Diaconis-Fill '90], [B-Ferrari '08]), we look for Markov chains on tilings that satisfy:

I. For each $k \geq 1$, the evolution of the bottom k rows $(\lambda^{(1)} \prec \lambda^{(2)} \prec \dots \prec \lambda^{(k)})$ is independent of the higher rows.

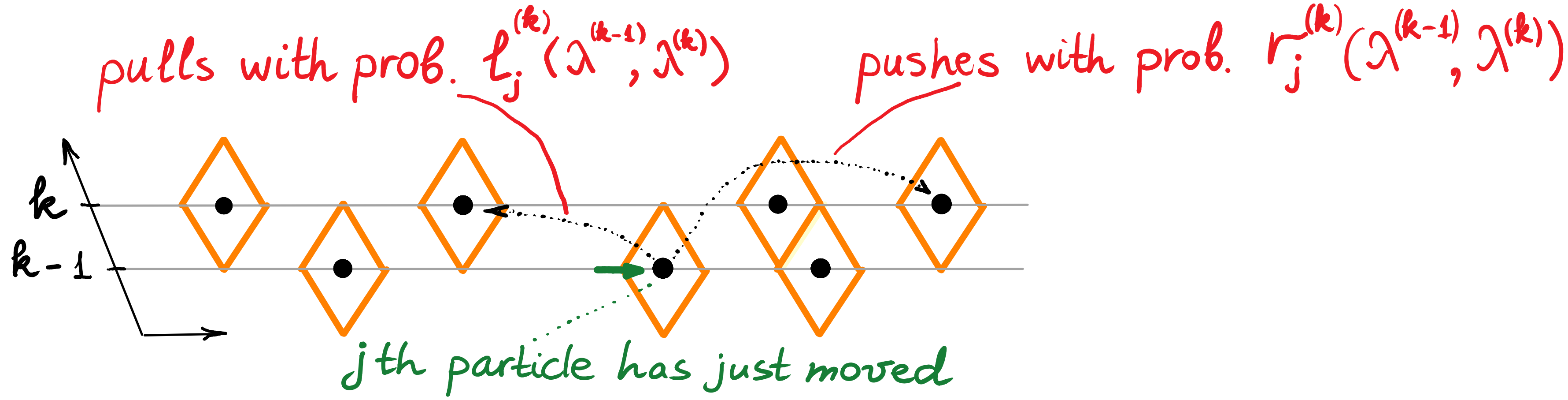
II. For each $k \geq 1$, the evolution preserves the Gibbs property on the bottom k rows:

$$m(\lambda^{(k)})\Lambda^k_{k-1}(\lambda^{(k)} \searrow \lambda^{(k-1)}) \cdots \Lambda^2_1(\lambda^{(2)} \searrow \lambda^{(1)}) \xrightarrow{\text{time } t} \tilde{m}(\lambda^{(k)})\Lambda^k_{k-1}(\lambda^{(k)} \searrow \lambda^{(k-1)}) \cdots \Lambda^2_1(\lambda^{(2)} \searrow \lambda^{(1)})$$

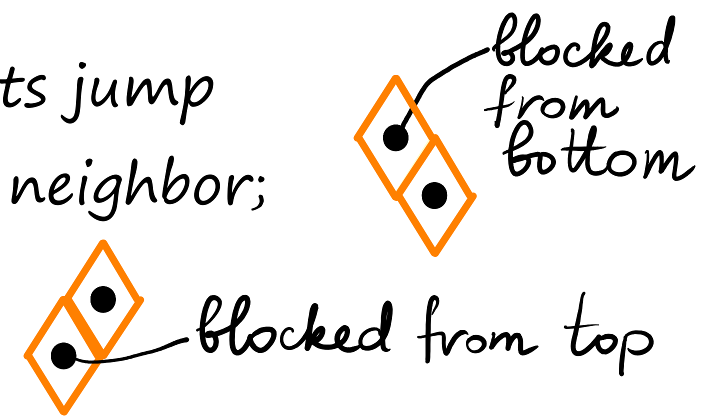
III. For each $k \geq 1$, the map $m \mapsto \tilde{m}$ is the time t evolution of the Markov chain with generator $L^{(k)}_{\text{Poisson}}$.

Nearest neighbor interaction

- Each particle jumps to the right by 1 independently, with exp. distributed waiting time; rate $w_j^{(k)}(\lambda^{(k-1)};\lambda^{(k)})$ for j-th particle on level k.
- A move of any particle may instantaneously trigger moves of its top-left (*pulling*) and top-right (*pushing*) neighbors.



\`No-nonsense': (a) If a particle is blocked from the bottom, its jump rate is 0, and when pushed it donates the move to its right neighbor; (b) If a particle is blocked from the top, $r_j = 1$.



Classification of nearest neighbor dynamics

Theorem [B-Petrov '13] A nearest neighbor Markov evolution satisfies I-III (independence of bottom rows, preservation of Gibbs, horizontal sections evolve according to $L^{(k)}_{Poisson}$) if and only if for any $k \geq 1$ and any $j \geq 0$ such that (j+1)st particle on level k is not blocked from the bottom,

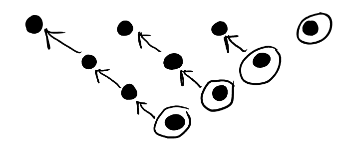
$$r^{(k)}_{j+1} + \ell^{(k)}_{j} + w^{(k)}_{j+1} = 1$$

no Vandermondes!

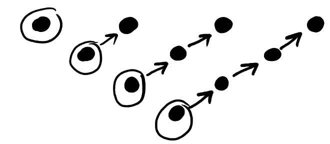
with nonexisting parameters at edges set to 0.

There are many solutions, all act the same on the Gibbs measures!

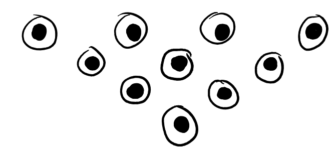
- $\ell_j \equiv 1,\ r_j \equiv 0,\ w_j = \begin{cases} 1, & j=1 \\ 0, & j>1 \end{cases}$ gives row RSK



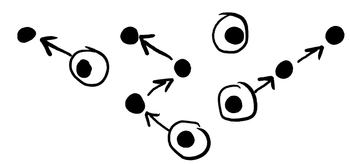
- $\ell_j \equiv 0,\ r_j \equiv 1,\ w_j = \begin{cases} 1, & j \text{ maximal} \\ 0, & \text{otherwise} \end{cases}$ gives column RSK



- $\ell_j = r_j \equiv 0,\ w_j \equiv 1$ gives push-block dynamics

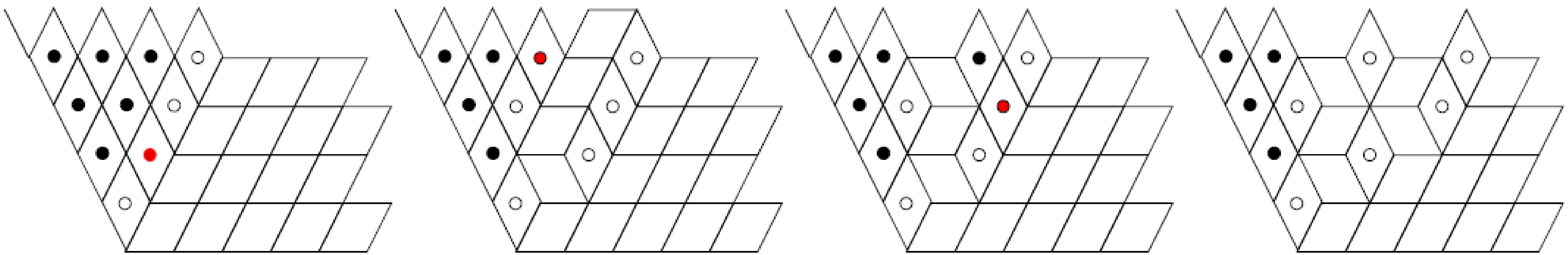


Many other possibilities, e.g.

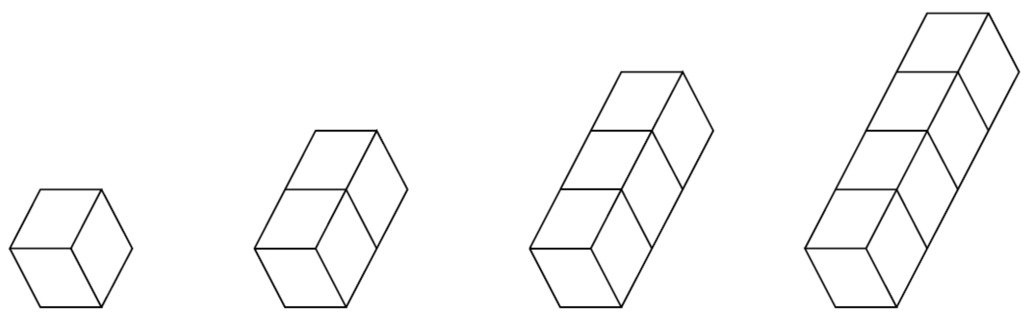


The push-block dynamics [B-Ferrari '08]

Each particle jumps to the right with rate 1. It is blocked by lower particles and it (short-range) pushes higher particles.



In 3d, this can be viewed as adding directed columns



[Column deposition - Animation]

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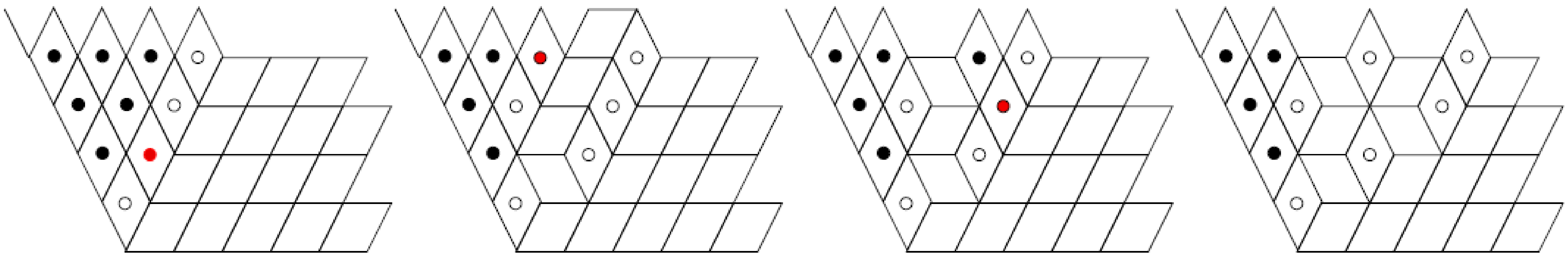
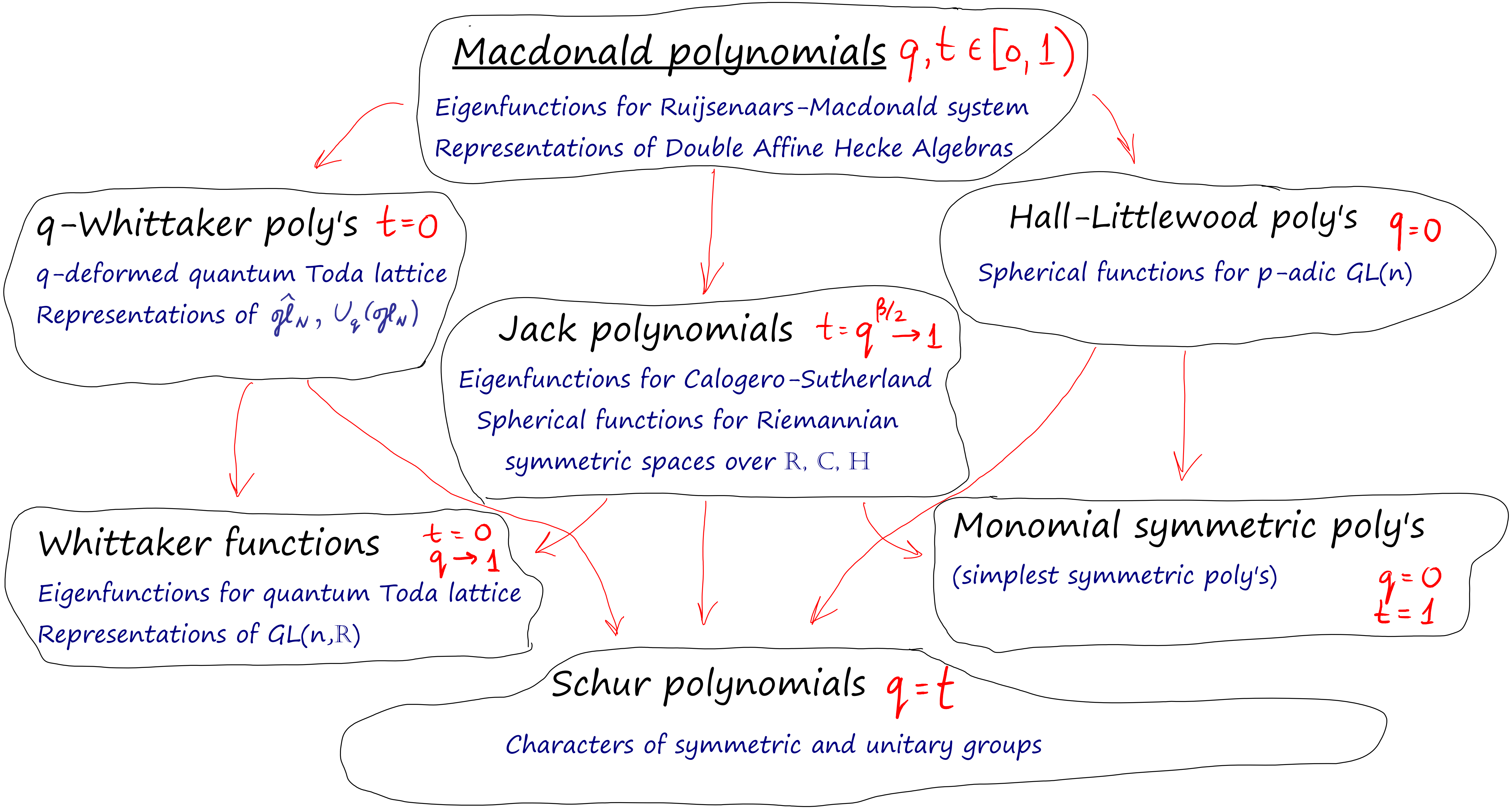
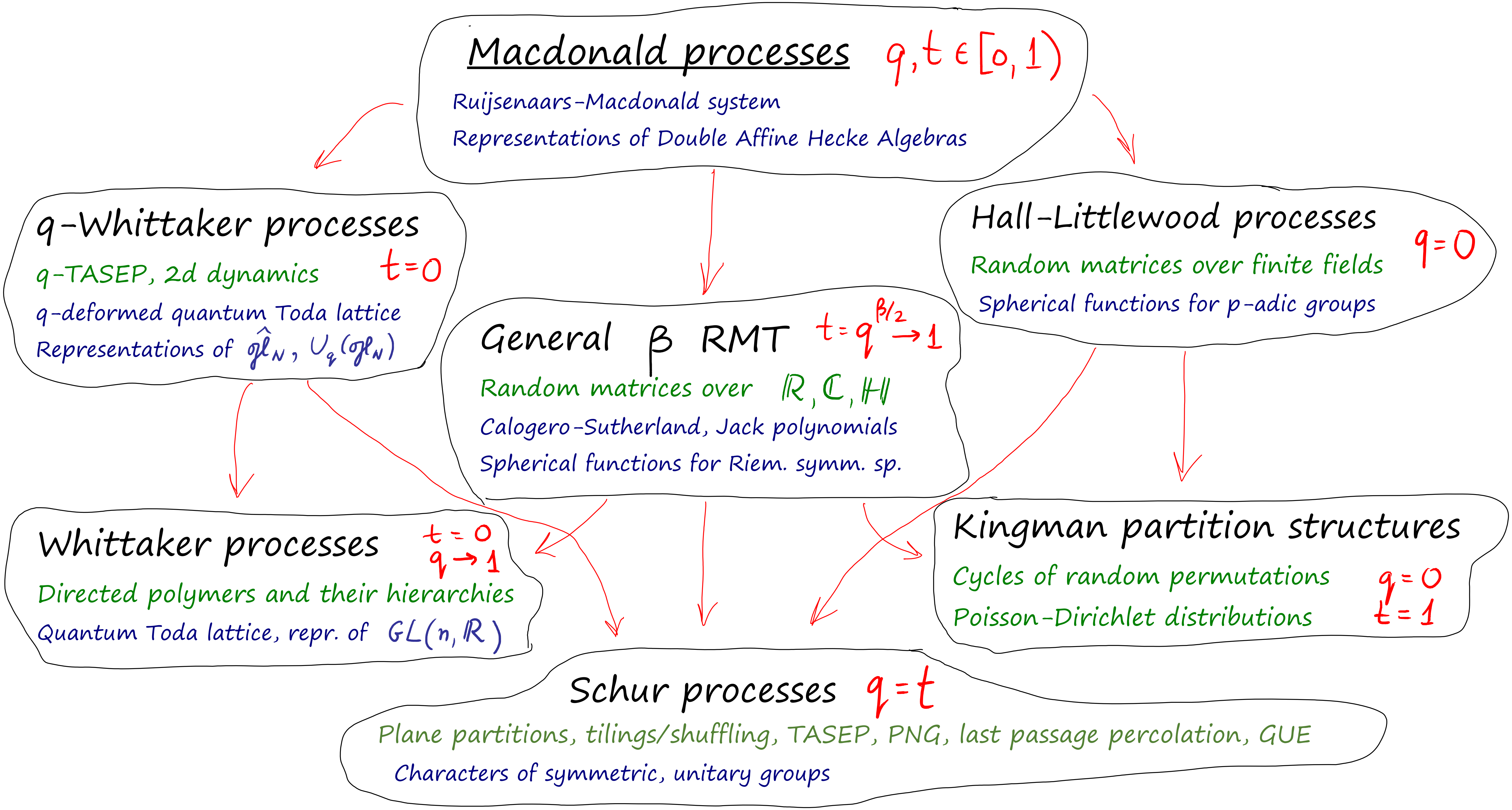
Each particle jumps to the right with rate 1. It is blocked by lower particles and it (short-range) pushes higher particles.



- Left-most particles form TASEP
- Right-most particles form PushTASEP

Previously studied asymptotics thus yields detailed information on large time behavior of these (2+1)d AKPZ and (1+1)d AKPZ models.


Macdonald polynomials $q, t \in [0,1)$
Eigenfunctions for Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker poly's $t=0$
q-deformed quantum Toda lattice
Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
Hall-Littlewood poly's $q=0$
Spherical functions for p-adic $GL(n)$
Jack polynomials $t=q^{\beta/2}\to 1$
Eigenfunctions for Calogero-Sutherland
Spherical functions for Riemannian
symmetric spaces over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$
Whittaker functions $t=0$ $q\to 1$
Eigenfunctions for quantum Toda lattice
Representations of $GL(n,\mathbb{R})$
Monomial symmetric poly's
(simplest symmetric poly's) $q=0$ $t=1$
Schur polynomials $q=t$
Characters of symmetric and unitary groups


Macdonald processes $q,t\in[0,1)$
Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker processes
$t=0$
q-TASEP, 2d dynamics
q-deformed quantum Toda lattice
Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
Hall-Littlewood processes
$q=0$
Random matrices over finite fields
Spherical functions for p-adic groups
General β RMT
$t=q^{\beta/2}\to 1$
Random matrices over $\mathbb{R},\mathbb{C},\mathbb{H}$
Calogero-Sutherland, Jack polynomials
Spherical functions for Riem. symm. sp.
Whittaker processes
$t=0$
$q\to 1$
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n,\mathbb{R})$
Kingman partition structures
Cycles of random permutations
Poisson-Dirichlet distributions
$q=0$
$t=1$
Schur processes $q=t$
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups

Macdonald polynomials $P_\lambda(x_1,\dots,x_N)\in\mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ labelled by partitions $\lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_N\geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$\mathcal{D}_1=\sum_{i=1}^{N}\Big(\prod_{a<b}(x_a-x_b)^{-1}\,T_{t,x_i}\prod_{a<b}(x_a-x_b)\Big)T_{q,x_i}=\sum_{i=1}^{N}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}\,T_{q,x_i}$$

with (generically) pairwise different eigenvalues $(T_q f)(z)=f(qz)$

$$\mathcal{D}_1P_\lambda=(q^{\lambda_1}t^{N-1}+q^{\lambda_2}t^{N-2}+\dots+q^{\lambda_N})P_\lambda\,.$$

Macdonald polynomials have many remarkable properties that include orthogonality, simple reproducing kernel (Cauchy identity), Pieri and branching rules, index/variable duality, simple higher order Macdonald difference operators that commute with $\mathcal{D}_1$, etc.

Single level distributions

As in the Schur case, one can define probability measures via

$$\prod_{i=1}^{N} e^{\gamma(x_i-1)} = \sum_{\mu=(\mu_1\geq\cdots\geq\mu_N\geq 0)} \mathrm{Prob}_\gamma\{\mu\} \cdot \frac{P_\mu(x_1,\dots,x_N)}{P_\mu(1,\dots,1)}.$$

These are time γ distributions of the Markov chain with jump rates

$$L^{(N)}_{\text{Poisson}}(\mu\to\nu) = \sum_\nu \varphi_{\nu/\mu} \cdot \frac{P_\nu(1,\dots,1)}{P_\mu(1,\dots,1)}$$

replace Vandermondes

with $\varphi_{\nu/\mu}$ given by the Pieri rule (they are 0 or 1 for Schur)

$$(x_1+\dots+x_N)\,P_\mu(x_1,\dots,x_N) = \sum_\nu \varphi_{\nu/\mu}\, P_\nu(x_1,\dots,x_N).$$

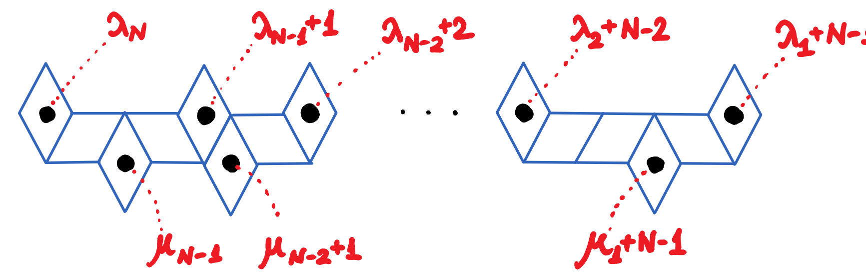
For $t=0$, $\varphi_{\mu+\vec{e}_j/\mu} = 1-q^{\mu_{j-1}-\mu_j}$.

This is a (q,t)-analog of the Dyson Brownian Motion.

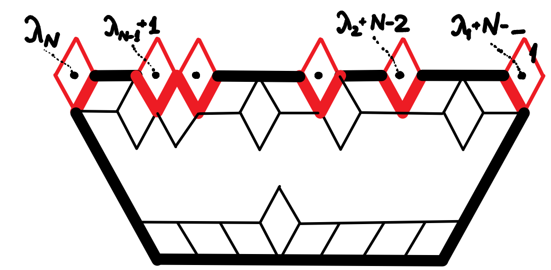
Representation theoretic object: Quantum Random Walk.

The (q,t)-Gibbs property

We define stochastic links Λ^N_{N-1} between N-tuples and $(N-1)$-tuples of integers using the branching rule



$$\frac{P_\lambda(x_1,\dots,x_{N-1},1)}{P_\lambda(1,\dots,1)} = \sum_{\mu\prec\lambda} \Lambda^N_{N-1}(\lambda\searrow\mu)\cdot\frac{P_\mu(x_1,\dots,x_{N-1})}{P_\mu(1,\dots,1)}.$$



Def. Random interlacing arrays $\lambda^{(1)}\prec\lambda^{(2)}\prec\dots\prec\lambda^{(N)}$ have the Macdonald-Gibbs property iff

$$\mathrm{Prob}\left\{(\lambda^{(1)},\dots,\lambda^{(N-1)})\,|\,\lambda^{(N)}\right\} = \Lambda^N_{N-1}(\lambda^{(N)}\searrow\lambda^{(N-1)})\,\Lambda^{N-1}_{N-2}(\lambda^{(N-1)}\searrow\lambda^{(N-2)})\dots\Lambda^2_1(\lambda^{(2)}\searrow\lambda^{(1)}).$$

$n!_q = 1\cdot(1+q)\cdot(1+q+q^2)\cdots(1+q+\dots+q^{n-1})$

For $t=0$ the links are $\Lambda^N_{N-1}(\lambda\downarrow\mu) = \dfrac{P_\mu(1,\dots,1)}{P_\lambda(1,\dots,1)}\cdot\displaystyle\prod_{i=1}^{N-1}\frac{(\lambda_i-\lambda_{i+1})!_q}{(\lambda_i-\mu_i)!_q\,(\mu_i-\lambda_{i+1})!_q}.$

Macdonald processes

An (ascending) Macdonald process is a distribution on $\lambda^{(1)} \prec \lambda^{(2)} \prec \lambda^{(3)} \prec \cdots$ that is (q,t)-Gibbs (once can also use $(a_1, a_2, \ldots)$ instead of $(1,1,\ldots)$).

Example 1: Decompositions of $\prod_{i=1}^{N} e^{\gamma(x_i - 1)}$ correspond to the \`Plancherel specialization' (consistency with Gibbs is nontrivial).

Example 2: $t = q^{\theta} \to 1$, $a_j = t^j$ for $j \geq 1$, \`principal specialization'.
Single level measures converge to general $\beta = 2\theta$ Jacobi ensembles

$$\mathrm{const} \cdot \prod_{i<j} |y_i - y_j|^{\beta} \prod_i y_i^{s_0} (1 - y_i)^{s_1} \, dy_i, \qquad y_i \in (0,1).$$

Example 3: Plancherel specialization, $t=0$. Leads to local 2d dynamics, q-TASEP, q-PushASEP, random polymers in $(1+1)$d.
Will be our focus.

Macdonald operators

Macdonald's q-difference operators diagonalized by P_λ are

$$\mathcal{D}^{(k)} = \sum_{I \subset \{1,\dots,N\}} \prod_{\substack{i\in I\\ j\notin I}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}, \qquad \mathcal{D}^{(k)} P_\lambda = e_k(q^{\lambda_1} t^{N-1}, \dots, q^{\lambda_N}) P_\lambda,$$

where $e_k(z_1, z_2, \dots) = \sum_{i_1<\dots<i_k} z_{i_1}\cdots z_{i_k}$. Using $\mathcal{D}\chi\big|_{x_j=1} = \sum_\lambda d_\lambda \operatorname{Prob}\{\lambda\} = \mathbb{E} d_\lambda$ with these operators gives many observables with explicit averages.

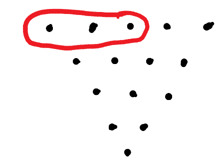
<u>Example 1:</u> For the Jacobi ensembles $\prod_{i<j} |y_i - y_j|^\beta \prod_i y_i^{s_0} (1-y_i)^{s_1}$ this gives averages of the powers sums $\sum_i y_i^k$ and of their products.

<u>Example 2:</u> For $t=0$ this gives averages of products of $q^{\lambda_N + \lambda_{N-1} + \dots + \lambda_{N-k+1}}$.

Integrals and scaling limits

For $t=0$ and Plancherel specialization (decomposition of $\prod_{i=1}^{N} e^{\gamma(x_i-1)}$), turning Macdonald operator $\mathcal{D}_k$ into a contour integral gives

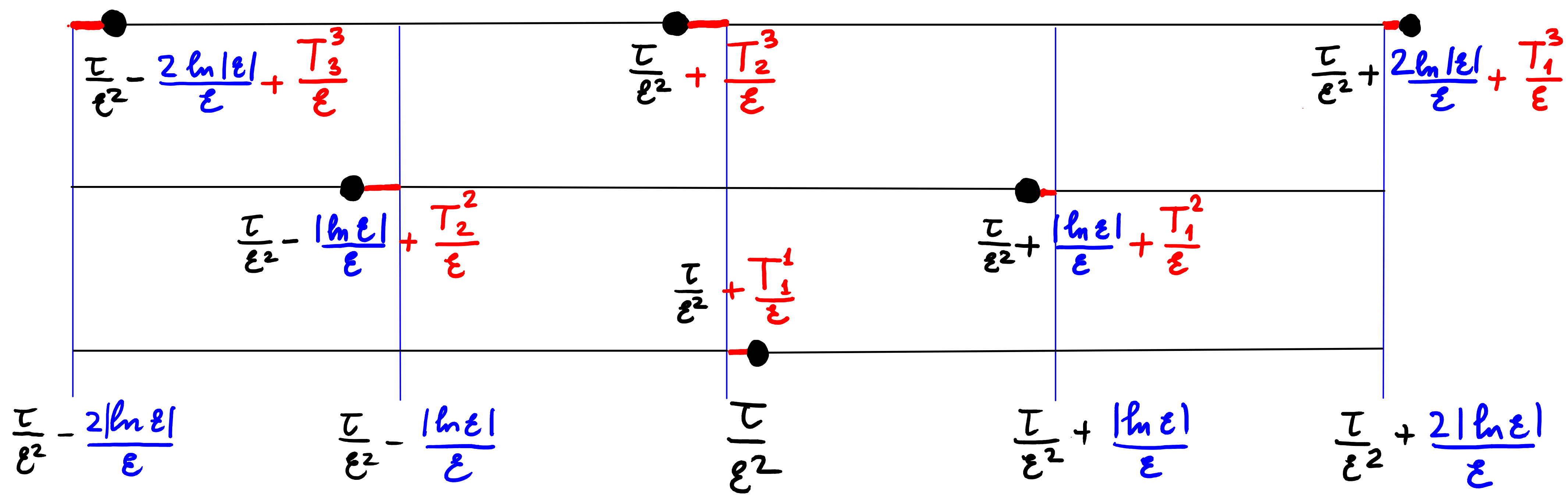
$$\mathbb{E}\, q^{\lambda_N^{(N)}+\ldots+\lambda_{N-k}^{(N)}} = \frac{(-1)^{\frac{k(k+1)}{2}}}{(2\pi i)^k\, k!} \oint_{\text{around } 1} \cdots \oint \prod_{1\le A<B\le k} (z_A - z_B)^2 \prod_{j=1}^{k} \frac{e^{(q-1)\gamma z_j}}{(1-z_j)^N} \frac{dz_j}{z_j^k}$$



$N=5$
$k=3$

The RHS has a clear limit as $q=e^{-\varepsilon} \to 1$, $\gamma = \text{const}\cdot\varepsilon^{-1}$, z_j's unchanged.
This leads to a LLN $\lambda_j^{(N)} \sim c_j^{(N)} \cdot \varepsilon^{-1}$ and Gaussian fluctuations of size $\varepsilon^{-1/2}$.

A less obvious limit is $q=e^{-\varepsilon} \to 1$, $\gamma = \tau\cdot\varepsilon^{-2}$, $z_j = 1+\varepsilon w_j$ for $1\le j\le k$.
Then the RHS behaves as $e^{-\tau k \varepsilon^{-1}} \cdot \varepsilon^{k(k-N)} \cdot$ finite integral.
This suggests the following scaling behavior:



<u>Theorem</u> [B-Corwin '11] As $q = e^{-\varepsilon} \to 1$, $\gamma = \tau \cdot \varepsilon^{-2}$, under the scaling

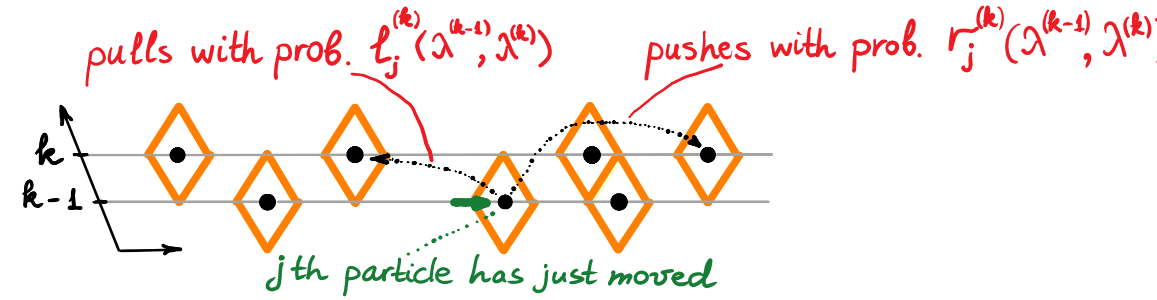
$$\lambda_j^{(N)} = \tau \varepsilon^{-2} - (N+1-2j)\frac{\ln \varepsilon}{\varepsilon} + T_j^N \varepsilon^{-1}$$

the t=0 Macdonald process with Plancherel specialization weakly converges to a probability distribution on real arrays $\{T_j^N\}$ (*the Whittaker process*).

Is there a probabilistic meaning behind the Whittaker process?

Back to Markov dynamics

The classification problem for the nearest neighbor Markov dynamics that preserve Gibbs measures and coincides with (q,t)-DBM on each level is (as for Schur) equivalent to a linear system of equations of the form [B-Petrov '13]



$$B_{j+1}^{(k)}\, r_{j+1}^{(k)} + B_j^{(k)}\, \ell_j^{(k)} + w_{j+1}^{(k)} = A_{j+1}^{(k)}$$

For t=0, the quantities $A_j^{(k)}$ and $B_j^{(k)}$ are local:

$$A_j^{(k)} = \frac{\left(1-q^{\lambda_{j-1}^{(k-1)}-\lambda_j^{(k)}}\right)\left(1-q^{\lambda_j^{(k)}-\lambda_{j+1}^{(k)}+1}\right)}{1-q^{\lambda_j^{(k)}-\lambda_j^{(k-1)}+1}}, \qquad B_j^{(k)} = \frac{\left(1-q^{\lambda_j^{(k-1)}-\lambda_{j+1}^{(k)}}\right)\left(1-q^{\lambda_{j-1}^{(k-1)}-\lambda_j^{(k-1)}+1}\right)}{1-q^{\lambda_j^{(k)}-\lambda_j^{(k-1)}+1}}.$$

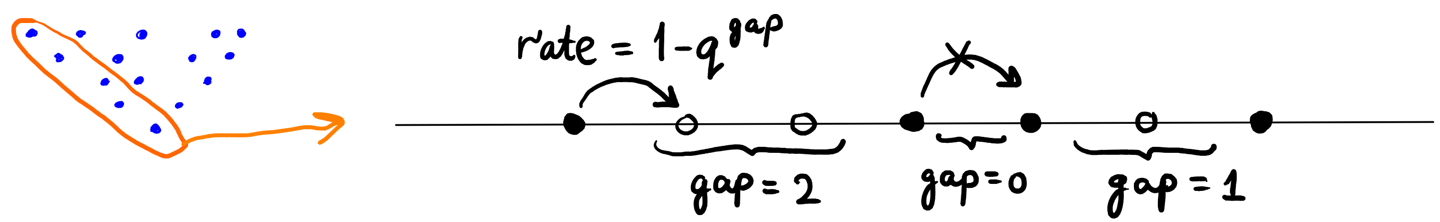
q-TASEP, q-PushTASEP, and 2d dynamics

There are many solutions. Imposing no pulling/pushing over distances >1 leads to the 2d local dynamics of [B-Corwin '11]:

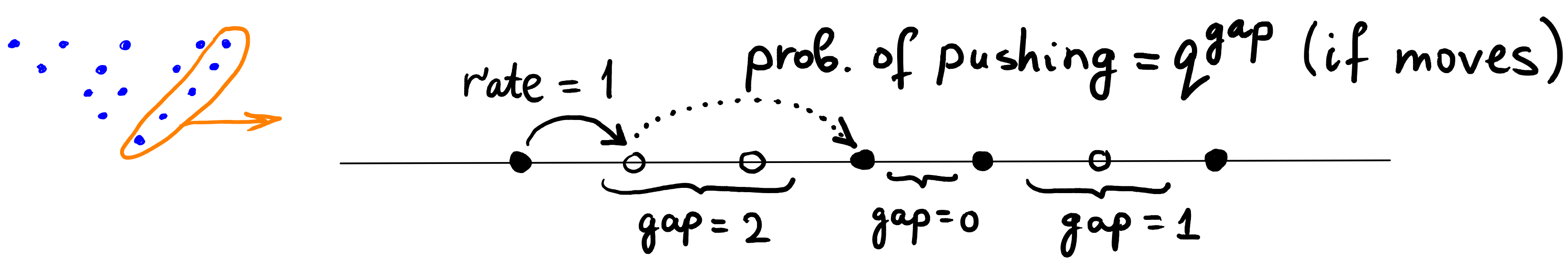
$$\ell_j = r_j \equiv 0, \qquad w_j^{(k)} = \frac{\left(1 - q^{\lambda_{j-1}^{(k-1)} - \lambda_j^{(k)}}\right)\left(1 - q^{\lambda_j^{(k)} - \lambda_{j+1}^{(k)} + 1}\right)}{1 - q^{\lambda_j^{(k)} - \lambda_j^{(k-1)} + 1}}.$$

Simulation

Projecting to left-most particles of each row yields q-TASEP:



Imposing almost sure jump propagation $\ell_j + r_j \equiv 1$ and $w_j = \begin{cases} 1, & j=1 \\ 0, & j>1 \end{cases}$
and further projecting to right-most particles yields q-PushTASEP:



Semi-discrete Brownian directed polymers

Whittaker scaling on q-PushTASEP (and q-TASEP) yields

$$dT_1^N = dB_N + e^{T_1^{N-1} - T_1^N} d\tau, \qquad N \geq 1,$$

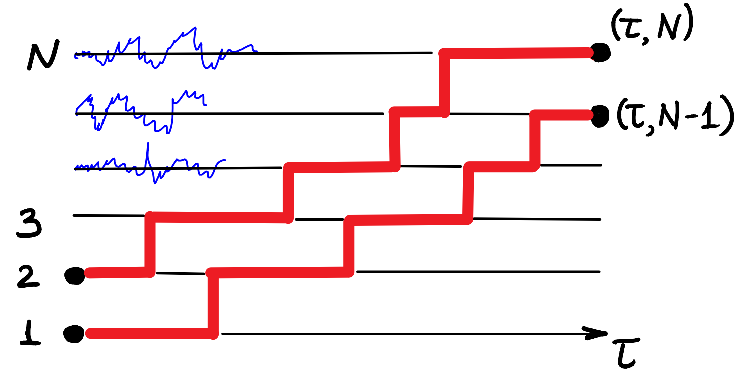
with independent Brownian motions $B_1, B_2, \dots$ (same for $\{-T_N^N\}_{N\geq 1}$).

Solving gives $T_1^N = \log \int_{0<s_1<\dots<s_{N-1}<\tau} e^{B_1(s_1) + (B_2(s_2) - B_2(s_1)) + \dots + (B_N(\tau) - B_N(s_{N-1}))} ds_1 \cdots ds_{N-1}$.

Theorem [O'Connell '09], [B-Corwin '11]

Lebesgue

$$T_1^N + \dots + T_k^N = \log \int \cdots \int e^{E(\phi_1) + \dots + E(\phi_k)} d\phi_1 \cdots d\phi_k$$



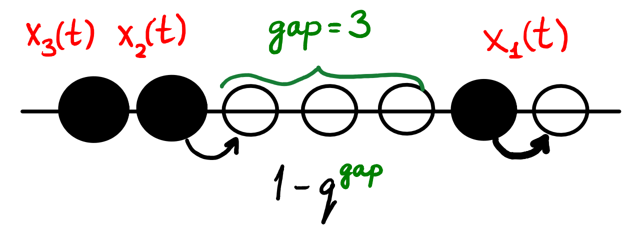
with integration over nonintersecting paths from (1,…,k) to (N-k+1,…,N). The measure is symmetric with respect to the flip $\{T_k^N \longleftrightarrow -T_{N-k+1}^N\}$.

$$E(\phi) = \int_\phi (\mathbb{R} \times \mathbb{Z})\text{-white noise}$$

$$= B_j(s_1) + (B_{j+1}(s_2) - B_{j+1}(s_1)) + \dots$$

q-TASEP moments

We now focus on left-most particles (q-TASEP) and wish to study the asymptotics as N gets large.

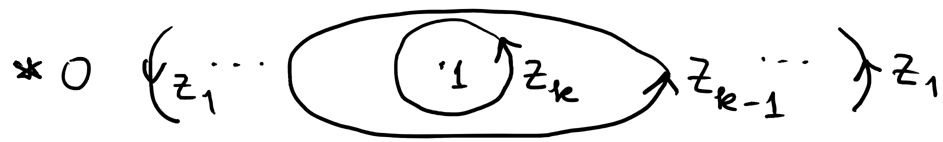


<u>Theorem</u> [B-Corwin '11], [B-C-Sasamoto '12], [B-C-Gorin-Shakirov '13]

For the q-TASEP with step initial data $\{x_n(0)=-n\}_{n\ge1}$

$$\mathbb{E}\, q^{(x_{N_1}(t)+N_1)+\dots+(x_{N_k}(t)+N_k)} = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint\cdots\oint \prod_{A<B} \frac{z_A-z_B}{z_A-qz_B} \prod_{j=1}^{k} \frac{e^{(q-1)tz_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j}$$

$(N_1 \ge N_2 \ge \dots \ge N_k)$



<u>Proof.</u> Consider the Macdonald process with Plancherel specialization and apply k first order Macdonald operators in $N_1, N_2, \dots, N_k$ variables. $\square$

Another proof via Quantum Integrable Systems will be given in Lecture 3.

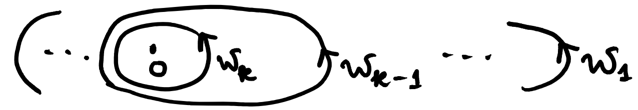
Polymer moments via nested integrals

By (formal) limit transitions:

For $$Z(N,\tau) = \int_{0<s_1<\dots<s_{N-1}<\tau} e^{B_1(s_1)+(B_2(s_2)-B_2(s_1))+\dots+(B_N(\tau)-B_N(s_{N-1}))} ds_1\cdots ds_{N-1}$$

$$\mathbb{E}\left[Z(N_1,\tau)\cdots Z(N_k,\tau)\right] = \frac{e^{\tau k/2}}{(2\pi i)^k}\oint\cdots\oint \prod_{1\le A<B\le k}\frac{w_A-w_B}{w_A-w_B-1}\prod_{j=1}^{k}\frac{e^{\tau w_j}}{w_j^k}\,dw_j$$

$N_1 \geq N_2 \geq \dots \geq N_k$



For $$Z(x,t) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\int :\exp:\left\{\int_0^t \dot{W}(s,b(s))\,ds\right\} db$$

Brownian bridge $b:[0,t]\to\mathbb{R}$, $b(0)=0$, $b(t)=x$

2d white noise

$$\frac{\partial Z}{\partial t} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} + \dot{W}Z \qquad \text{SHE}$$

$$\frac{\partial \ln Z}{\partial t} = \frac{1}{2}\frac{\partial^2 \ln Z}{\partial x^2} + \left(\frac{\partial \ln Z}{\partial x}\right)^2 + \dot{W} \qquad \text{KPZ}$$

$$\mathbb{E}\left[Z(x_1,t)\cdots Z(x_k,t)\right] = \int_{\alpha_1-i\infty}^{\alpha_1+i\infty} dz_1 \int_{\alpha_2-i\infty}^{\alpha_2+i\infty} dz_2 \cdots \int_{\alpha_k-i\infty}^{\alpha_k+i\infty} \prod_{1\le A<B\le k}\frac{z_A-z_B}{z_A-z_B-1}\prod_{j=1}^{k} e^{\frac{t}{2}z_j^2+x_j z_j}$$

$x_1 \leq x_2 \leq \dots \leq x_k$

$\alpha_1 > \alpha_2 + 1 > \dots > \alpha_k + (k-1)$

Is this sufficient for determining the distributions of Z's?

Intermittency

Polymer partition functions Z are intermittent. Higher moments are dominated by higher peaks and do not determine the distrib. This is measured by moment Lyapunov exponents $\gamma_p = \lim_{t\to\infty} \frac{\ln \mathbb{E} Z^p(t)}{t}$.

$\frac{\gamma_p}{p} \neq \text{const}$ means intermittency [Zeldovitch et al. '87].

By steepest descent in nested integrals one shows:

<u>Semi-discrete</u>: [B-Corwin '12]

$\gamma_p = H_p(z_c)$, where (for $N=\tau$) z_c is the crit. point of

$$H_p(z) = \frac{p^2}{2} + pz - \log\left(\frac{\Gamma(z+p)}{\Gamma(z)}\right) \text{ on } (0,+\infty).$$

<u>Continuous</u>: [Kardar '87], [Bertini-Cancrini '95]

$$\gamma_p = \frac{p^3 - p}{24}.$$

The speed of growth of Lyapunov exp's does not predict fluctuation exponents!

Replica trick

In its simplest incarnation, ignoring intermittency, replica trick analytically continues moments off positive integers and uses

$$\log Z = \lim_{p\to 0} \frac{Z^p - 1}{p}, \qquad \lim_{t\to\infty} \frac{\ln Z}{t} = \lim_{p\to 0} \lim_{t\to\infty} \frac{1}{t} \frac{e^{t\gamma_p}-1}{p} = \lim_{p\to 0} \frac{\gamma_p}{p}$$

to predict the almost sure behavior. This gives correct LLN values:

Semi-discrete: $\lim_{p\to 0} \frac{1}{p}\left(\frac{p^2}{2} + pz - \log\frac{\Gamma(z+p)}{\Gamma(z)}\right) = z - (\log\Gamma(z))'$, take value at crit. point on $(0,+\infty)$

Proved: [O'Connell–Yor '01], [Moriarty–O'Connell '07]

Continuous: $\lim_{p\to 0} \frac{1}{p}\cdot\frac{p^3 - p}{24} = -\frac{1}{24}.$

Proved: [Amir–Corwin–Quastel '10], [Sasamoto–Spohn '10]

More elaborate treatment of moments gives limiting fluctuations [Dotsenko '10+], [Calabrese–Le Doussal–Rosso '10+]. WHY?

q-TASEP moments and contour deformation

The distribution of q^{x_n} for q-TASEP particles is NOT intermittent.

We can find the distribution and then take the limit to polymers.

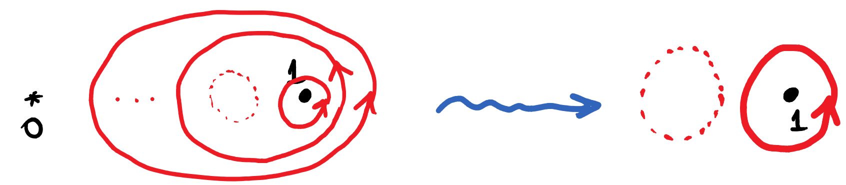
But nested contours are not suited for very large moments.

Lemma

$$\frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \underbrace{\oint \cdots \oint}_{\text{nested}} \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - q z_B} \, \frac{f(z_1)\cdots f(z_k)}{z_1\cdots z_k} \, dz_1 \cdots dz_k \qquad f(z) = \frac{e^{(q-1)tz}}{(1-z)^N}$$

$$= k!_q \sum_{\substack{\lambda=(\lambda_1\ge\ldots\ge\lambda_\ell\ge 0)\\ \lambda_1+\ldots+\lambda_\ell=k}} \frac{1}{m_1!\, m_2! \cdots} \, \frac{(1-q)^k}{(2\pi i)^\ell} \underbrace{\oint \cdots \oint}_{\text{small}} \det\left[\frac{1}{w_i q^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell} \prod_{j=1}^{\ell} f(w_j) f(q w_j) \cdots f(q^{\ell-1} w_j)\, dw_j$$

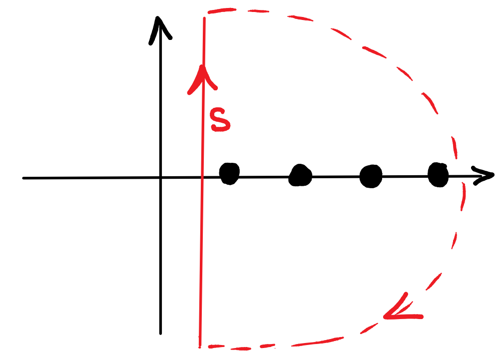
$$\lambda = 1^{m_1} 2^{m_2} \ldots$$



This formula plays a key role in spectral analysis of Quantum Integrable Systems in Lecture 3. The dets are similar to inverse squared normes of Bethe eigenstates.

Laplace transforms

It is convenient now to take the generating function $\sum_{k\ge 0} \mathbb{E}\left(q^{x_N}\right)^k \frac{\zeta^k}{k!_q}$.
Replace the sum over ordered cluster sizes by that over unordered unrestricted integers $n_1, n_2, \ldots$ (removes the combinatorial factor), and use the Mellin-Barnes transform



$$\sum_{n\ge 1} g(q^n)\,\zeta^n = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \underbrace{\Gamma(-s)\,\Gamma(1+s)}_{=\frac{\pi}{\sin \pi s}}\,(-\zeta)^s\, g(q^s)\,ds$$

The result admits direct term-wise limit to polymers:

$$\mathbb{E}\, e^{-u Z(N,\tau)} = 1 + \sum_{\ell\ge 0}\frac{1}{\ell!\,(2\pi i)^{2\ell}} \oint_{|v_j|=1/4}\cdots\oint dv_1\cdots dv_\ell \int_{3/4-i\infty}^{3/4+i\infty}\cdots\int ds_1\cdots ds_\ell$$

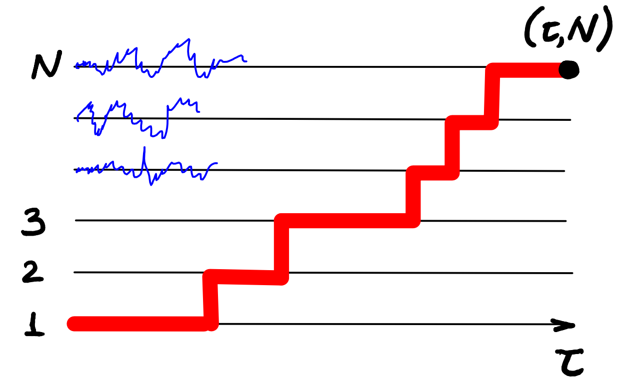
$$\times \prod_{j=1}^{\ell}\frac{\pi}{\sin \pi s_j}\, e^{\Phi(s_j+v_j)-\Phi(v_j)} \cdot \det\left[\frac{1}{s_i+v_i-v_j}\right]_{i,j=1}^{\ell}$$

$$\Phi(z) = \frac{\tau}{2} z^2 + z\cdot \ln u - N \ln \Gamma(z)$$

Limit theorem

Theorem [B-Corwin '11, B-Corwin-Ferrari '12] For any $\varkappa > 0$

$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{Z(N,\varkappa N) - f_\varkappa \cdot N}{g_\varkappa \cdot N^{1/3}} \le r \right\} = F_{GUE}(r)$$



The proof is by steepest descent analysis of the last expression.

The Tracy-Widom GUE distribution arises as

$$F_{GUE}(g_\varkappa r) = 1 + \sum_{\ell \ge 1} \frac{1}{\ell!\,(2\pi i)^{2\ell}} \int \cdots \int da_1 \cdots da_\ell \int \cdots \int db_1 \cdots db_\ell$$

$$\times \prod_{j=1}^{\ell} \frac{1}{a_j - b_j} \, \frac{\exp\left(-\frac{g_\varkappa^3}{3} a_j^3 + r a_j\right)}{\exp\left(-\frac{g_\varkappa^3}{3} b_j^3 + r a_j\right)} \cdot \det\left[\frac{1}{b_i - a_j}\right]_{i,j=1}^{\ell}.$$

Back to the replica trick

q-Laplace transform for q-TASEP = Generating series of q-TASEP moments

converges

converge

series diverges

Laplace transform of polymer partition function ⋯⋯ Generating series of polymer moments

replica trick

The bona fide argument on the q-level is the only currently available explanation of why the replica trick works in this case.

This will be extended in Lecture 3.

Macdonald processes $q, t \in [0,1)$

Ruijsenaars–Macdonald system

Representations of Double Affine Hecke Algebras

q-Whittaker processes $t=0$

q-TASEP, 2d dynamics

q-deformed quantum Toda lattice

Representations of $\widehat{gl}_N$, $U_q(gl_N)$

just discussed

General β RMT $t = q^{\beta/2} \to 1$

Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$

Calogero–Sutherland, Jack polynomials

Spherical functions for Riem. symm. sp.

Hall–Littlewood processes $q=0$

Random matrices over finite fields

Spherical functions for p-adic groups

Whittaker processes $t=0$, $q \to 1$

Directed polymers and their hierarchies

Quantum Toda lattice, repr. of $GL(n, \mathbb{R})$

Kingman partition structures

Cycles of random permutations $q=0$

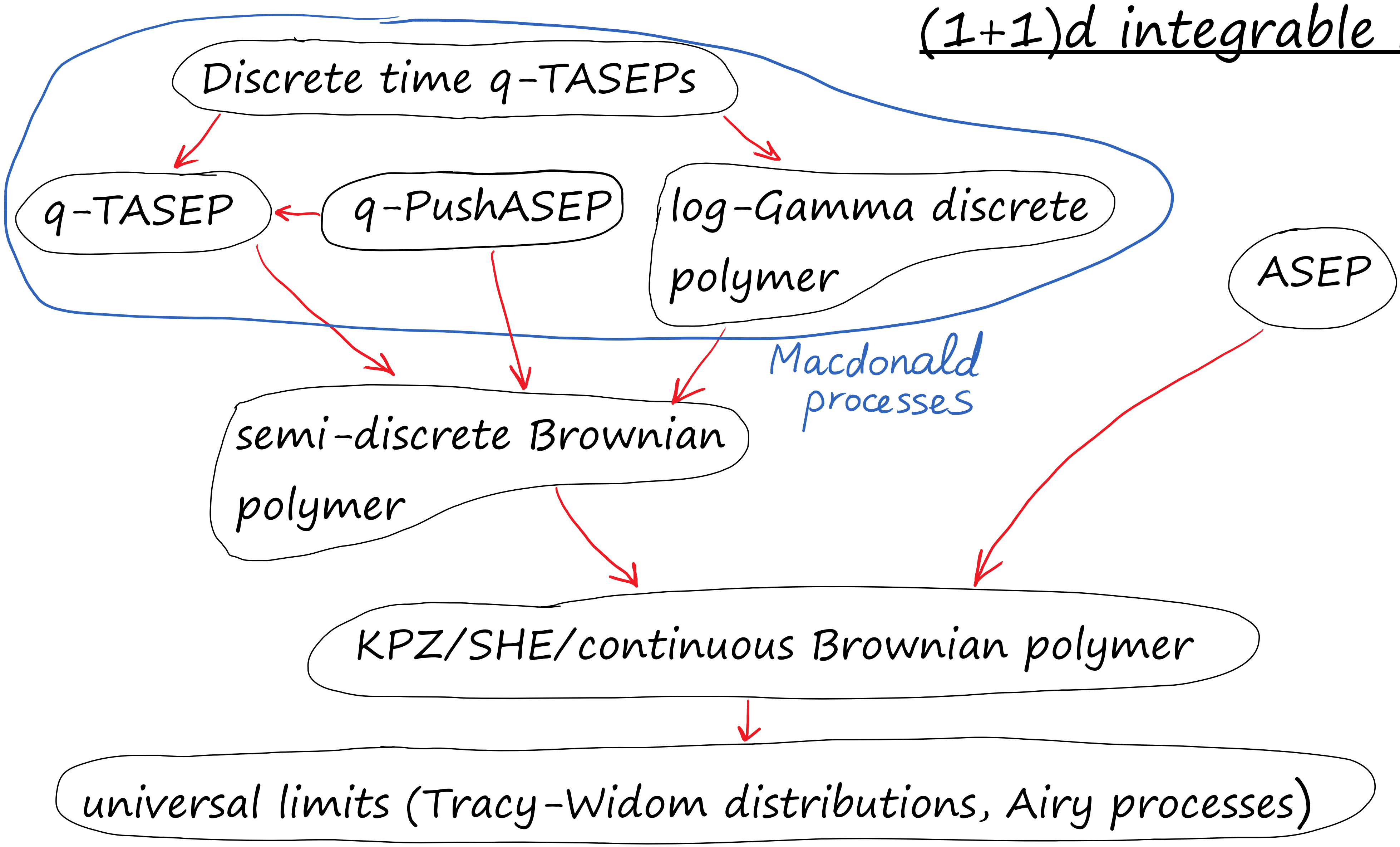
Poisson–Dirichlet distributions $t=1$

Schur processes $q=t$

Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE

Characters of symmetric, unitary groups

(1+1)d integrable KPZ systems



Aiming at accessing other integrable KPZ systems and more general initial conditions, Lecture 3 will present a different approach.