Spectral geometry on metric graphs

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Outline

Table of Contents

Spectral geometry

Say that (M, g) is a compact Riemannian manifold and Δ_g is its Laplace-Baltrami operator. Then Δ_g is self adjoint with a complete set of eigenfunctions $\{f_n\}_{n\in\mathbb{N}}$ with real non-negative eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 ... \nearrow \infty.$$

Spectral geometry

- Given such (M, g) can we compute the spectrum (eigenvalues)? Explicitly? Implicitly?
- Can we describe the dependence of the spectrum in g? in M?
- Are there properties of the eigenfunctions that we can measure and relate to g or to M? For example, nodal count.
- Kac 66' - "Can one hear the shape of a drum?". Can we deduce M or g from the spectrum?
- Can we deduce M and g using information obtained from $\{f_n\}_{n\in\mathbb{N}}$?

Metric graphs

Given a finite (discrete) graph $\Gamma(\mathcal{E},\mathcal{V})$, and a choice of $\vec{l}\in\mathbb{R}_{+}^{E}$ we denote the metric graph $\Gamma_{\vec{l}}$, such that every edge e has length l_e.

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What is the difference between a metric graph and a weighted discrete graph?

Metric graphs

The **Laplacian** Δ on $\Gamma_{\vec{l}}$ is defined, edgewise, by:

$$(\Delta f)\,|_e = -\left(\frac{d}{dx}\right)^2 f|_e.$$

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Neumann (Kirchhoff) vertex conditions at a vertex v:

1. Continuity of f at v.
2. $\sum_{e\sim v} \partial_e f\,(v) = 0$.

Spectral geometry on metric graphs

Given $\Gamma_{\vec{l}}$ with Δ and Neumann vertex conditions we get a complete family of eigenfunctions $\{f_n\}_{n\in\mathbb{N}}$ and their eigenvalues

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We refer to these as the spectrum and eigenfunctions of $\Gamma_{\vec{l}}$.
We may now ask about their Γ and $\vec{l}$ dependence.

Table of Contents

Examples for spectral results

Implicit calculation of the spectrum:

- von Below 85' :
  Secular function $F(k)$ whose zeros are the spectrum of $\Gamma_{\vec{l}}$.
- Kottos, Smilansky 97' :
  Exact trace formula for the spectral density

$$\sum_{n\in\mathbb{N}} \delta_{k_n} + \delta_{-k_n} = \frac{L}{\pi} + \lim_{\epsilon\to+0} \sum_{p} \sum_{r=1}^{\infty} \frac{L_p}{\pi} A_p^r \cos\left(kL_p + \phi_p\right) e^{-L_p\epsilon}.$$

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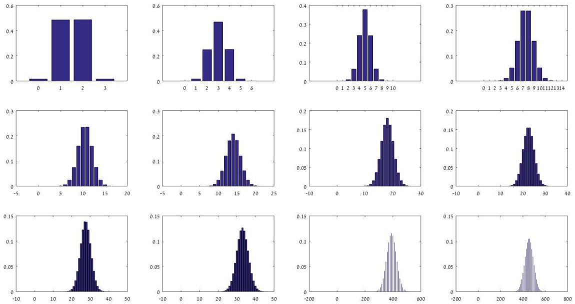
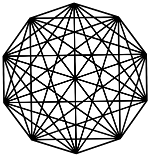
Recent result:

- Kurasov, Sarnak (2020):
  The spectral density of a metric graph with incommensurate lengths is an exotic positive crystalline measure. It is a Fourier quasi-crystal which does not contain any Dirac comb.
  They answered several open questions.

Results regarding eigenfunctions





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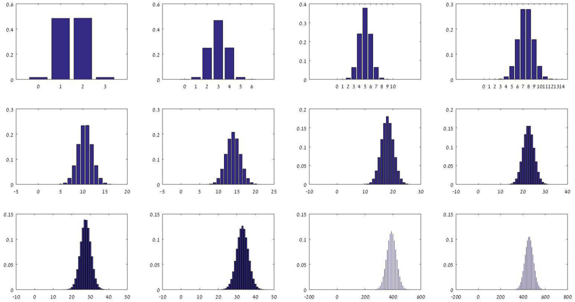

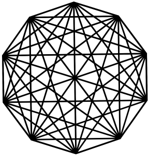

On a universal behaviour of the nodal and Neumann counts for metric graphs -
IAS Analysis seminar, October 12th at 16:30.

Table of Contents

Quantum graphs and quantum chaos

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I intend to work on this conjecture during my IAS period.

Table of Contents

The secular equation

Given a (discrete) graph Γ with E edges, there is a corresponding polynomial $P_\Gamma : \mathbb{C}^E \to \mathbb{C}$ such that for any choice of edge lengths $\vec{l} = (l_1, l_2 ... l_E)$,

$$\boxed{k^2 \text{ is an eigenvalue of } \Gamma_{\vec{l}} \iff P_\Gamma\left(e^{ikl_1}, e^{ikl_2}, ..., e^{ikl_E}\right) = 0}$$

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Consider the torus $\mathbb{T}^E := \mathbb{R}^E / 2\pi\mathbb{Z}^E$. The **Secular manifold** of Γ is the zero set

$$\Sigma := \left\{\vec{\kappa} \in \mathbb{T}^E \ : \ P_\Gamma\left(e^{i\kappa_1}, e^{i\kappa_2}, ..., e^{i\kappa_E}\right) = 0\right\}.$$

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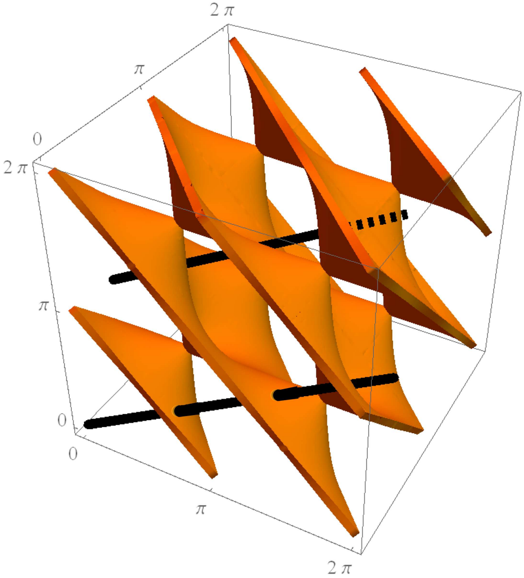
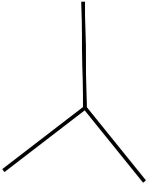
Consider the linear flow $\varphi_{\vec{l}} : \mathbb{R} \to \mathbb{T}^E$ given by $k \mapsto k\vec{l}\; mod\, 2\pi$. Then the secular equation is

$$\boxed{k^2 \text{ is an eigenvalue of } \Gamma_{\vec{l}} \iff \varphi_{\vec{l}}(k) \in \Sigma}$$

The Secular Manifold





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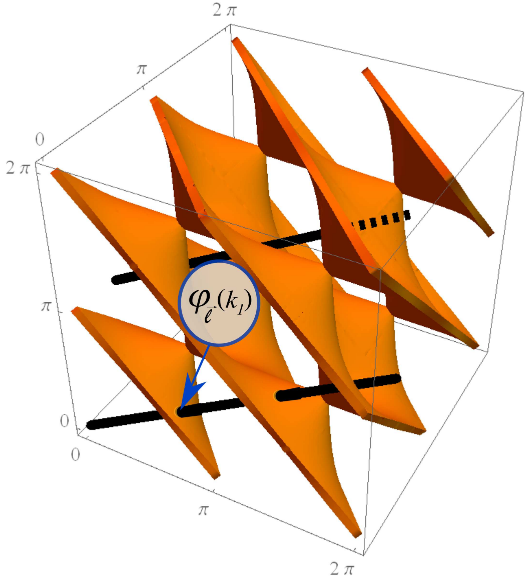

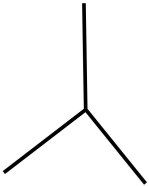

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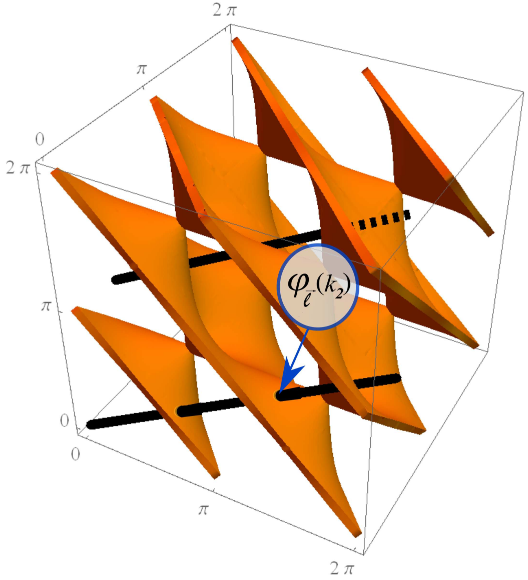

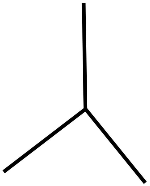

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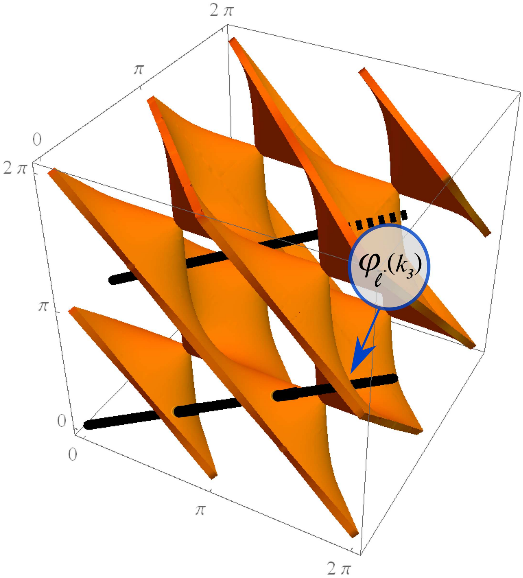

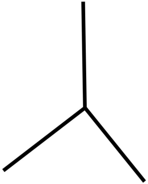

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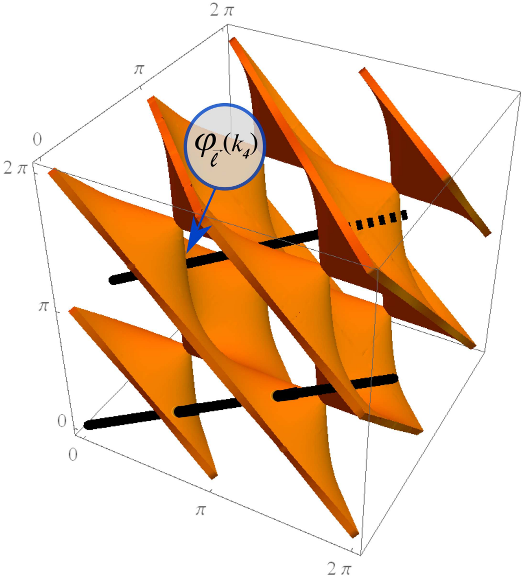

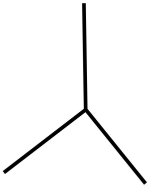

Ergodicity

Theorem (Barra, Gaspard '00; Berkolaiko, Winn '10; Colin de Verdière '15)

Let $\Gamma_{\vec{l}}$ *be a metric graph with incommensurate edge lengths and let* $\left\{k_n^2\right\}_{n\in\mathbb{N}}$ *be its spectrum. Then the sequence* $\varphi_{\vec{l}}(k_n)$ *is equidistributed on* Σ *according to a given measure* $\mu_{\vec{l}}$*.*

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The geometry of Σ now plays an important role.

The geometry of Σ

- Berkolaiko and Liu 17', A. (PhD thesis) -
  On the number of connected components of the regular part of Σ.
- Colin de Verdière 15' -
  A conjecture regarding the irreducibility of Σ.
  Proven by Kurasov and Sarnak (yet to be published).

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Open question -
What can we say about Σ in the limit of large graphs?
Is it becomes “flat” and in what rate?

- Quantum unique ergodicity.
- Level spacing statistics.

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