Effective density of Unipotent orbits

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Analysis & Beyond

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based on j/w Margulis, Mohammadi, Shah

$G =$ Linear algebraic gp $/\mathbb{R}$ [e.g. $SL_n(\mathbb{R})$, $SO_{n,m}(\mathbb{R})$]

Def 1: connected $U < G$ is unipotent if $\forall u \in U$ 1 is only complex eigenvalue of u.

Def 2: $U < G$ is horospheric if $\exists a \in G$ s.t. $U = \{g \in G : a^{-n} g a^{n} \longrightarrow e\}$ as $n \to \infty$

Proposition: Any horospheric gp is unipotent.

Examples: • $\{u_t\} = \left\{\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\right\}_{t\in\mathbb{R}}$ is horospheric in $SL_2(\mathbb{R})$

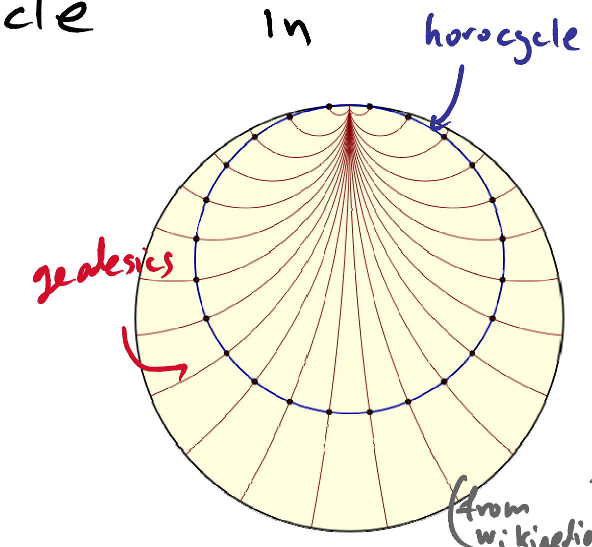
• $\left\{\begin{pmatrix} 1 & t & s \\ & 1 & r \\ & & 1 \end{pmatrix}\right\}_{t,s,r\in\mathbb{R}}$, $\left\{\begin{pmatrix} 1 & t & s \\ & 1 & 0 \\ & & 1 \end{pmatrix}\right\}_{t,s\in\mathbb{R}}$ are horospheric in $SL_3(\mathbb{R})$

• $\left\{\begin{pmatrix} 1 & t & \\ & 1 & \\ & & 1 \end{pmatrix}\right\}_{t\in\mathbb{R}}$, $\left\{\begin{pmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{pmatrix}\right\}_{t\in\mathbb{R}}$, $\left\{\begin{pmatrix} 1 & t & s \\ & 1 & t \\ & & 1 \end{pmatrix}\right\}_{t,s}$

are unipotent, but not horospheric in $SL_3(\mathbb{R})$.

$u_t = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ orbits in $SL_2(\mathbb{R})/\Gamma$, Γ lattice* are classical object: project to horocycles on $\mathbb{H}/\Gamma$

Thm (Hedlund, 30's): any horocycle in $\mathbb{H}/\Gamma$ ($\Gamma < SL_2(\mathbb{R})$ lattice) is either periodic or dense



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*lattice – gp $\Gamma < G$ s.t. Γ discrete, and G/Γ has finite volume.

Should be compared to behavior of $g_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$ orbits on $SL_2(\mathbb{R})/\Gamma$, which project to geodesics on $\mathbb{H}/\Gamma$:

Thm (folklore?): $\forall \alpha \in [0,2]$ $\exists x \in SL_2(\mathbb{R})/\Gamma$ s.t.

$$\dim_H \overline{\{g_t . x\}_{t \geq 0}} = 1 + \alpha$$

Note: by erogicity, for a.e. x $\{g_t . x\}_{t \geq 0}$ is dense

key pt: we want control on <u>all</u> orbits.

- Motivated by works of Dani, Furstenberg, Margulis
  Raghunathan conjectured that the rigidity properties of horocycle flow are a general feature of unipotent flows.
- Conj. established by Ratner, important special cases proved earlier by Margulis, Dani-Margulis

Def: X space, L acts on X. $x \in X$ is L-periodic if $\mathrm{Stab}_L(x)$ has finite covolume in L.

Thm [Ratner, 1991]: let G be Lie gp, $H < G$ gen. by Ad-unipotent one parameter gps, $\Gamma < G$ a lattice.

For any $x \in G/\Gamma$, $\overline{H.x}$ is <u>nice</u>:

$\exists L < G$ s.t. $\overline{H.x}$ is a periodic L-orbit.

(orbit closure rigidity)

Some Prior partial results:

- case of H horospherical established by Dani (81)

- $G = SL(3,\mathbb{R})$, $H = SO(2,1)$ $(\Gamma = SL(3,\mathbb{Z}))$

  $H.x$ is either periodic or unbounded (Margulis, c.85)

  implies Oppenheim Conj (Oppenheim, c. 30's)

  $H.x$ is either periodic or dense (Dani-Margulis, 89)

- $G = SL(3,\mathbb{R})$, $H = \left\{ \begin{pmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{pmatrix} \right\}_t$:

  $\overline{H.x}$ classified by Dani-Margulis, 90

- $G = SO(n,1)$, H 1-param unipotent

  dealt by Shah concurrently with Ratner

Note: for $U = \{u_t\}$ 1-param unipotent gp Ratner proved more than $\overline{U.x}$ nice:

Thm (Ratner): Let G be Lie gp, $\Gamma < G$ lattice, $U = \{u_t\}$ unipotent, $x \in G/\Gamma$. Then $\exists L < G$ s.t. $\overline{U.x} = L.x$ is periodic and for any $f \in C_c(G/\Gamma)$

$$\frac{1}{T}\int_0^T f(u_t.x)\,dt \longrightarrow \int_{L/\mathrm{Stab}} f(\ell.x)\,dm_L(\ell)$$

Key part of pf: classifying u_t-inv and ergodic probability measures on G/Γ.

Above thms extended to S-arithmetic context by Ratner and Margulis-Tomanov, with numerous applications.

Effective / quantitative results.

- For U a horospheric subgp, density & equidistribution of individual orbits are understood even quantatively (Sarnak 81, Burger 90, Flaminio-Forni, Strombergsson, Sarnak-Ubis, ...) e.g.

Thm (Strombergsson): Suppose $x \in SL_2(\mathbb{R})/\Gamma$, $f \in C_c(SL_2/\Gamma)$

$$\frac{1}{T}\int_0^T f(u_t.x)\,dt = \int_{SL_2(\mathbb{R})/\Gamma} f + O_f\left(r^{-1/2}\log^3 r + |r^{-s_1'}| + |T^{-s_1}|\right)$$

with

- $\lambda_1 = s_1(1-s_1)$ the smallest nontrivial eigenvalue of Δ on $\mathbb{H}/\Gamma$
- $\lambda_1' = s_1'(1-s_1')$ the smallest nontrivial non cuspidal eigenvalue of Δ on $\mathbb{H}/\Gamma$ (if exists)
- $r = T e^{-\mathrm{dist}_{\mathbb{H}/\Gamma}(g_{-t}\cdot x,\, p)}$ p fixed, $t = \log T$

- If G is semi-simple real algebraic gp def $/\mathbb{Q}$

$\Gamma < G(\mathbb{Q})$ congruence lattice, $H < G$ semisimple [with $C_G(H)$ finite, no compact factors] and $H.x_i$ is a sequence of H-periodic orbits

Thm [Einsiedler-Margulis-Venkatesh]: Let $H.x_0$ be periodic, $\mathrm{vol}(G/\Gamma) \leq V < \mathrm{vol}(H.x_0)$. Then there is a semisimple $L \leq G$, $L \geq H$ so that $L.x_0$ is periodic of $\mathrm{vol} \leq V$ and

$$\left| \int_{H.x_0} f - \int_{L.x_0} f \right| \ll c_f \cdot V^{-\delta} \qquad \forall f \in C_c(G/\Gamma)$$

Further work by Einsiedler, Margulis, Mohammadi, Venkatesh allows H to vary, G to be S-arithmetic.

Step -1: What is $\mathrm{vol}(H.x)$ when H is not fixed?

Recipe: Fix Ω open, relatively compact nbhd of $e \in G$.

$$\widehat{\mathrm{vol}}(H.x) = \frac{m_H(H/\mathrm{stab}_x H)}{m_H(H \cap \Omega)}$$

For G semisimple, U not horospherical $x \in G/\Gamma$ "generic"

quantitative understanding of density of $B_T^U . x$ seems harder.

Def: a real algebraic group H is said to be of class $\mathcal{H}$ if $H(\mathbb{C})$ is generated by unipotents

examples: $SL_n(\mathbb{R}) \times SU_n(\mathbb{R})$, $SL_n(\mathbb{R}) \ltimes \mathbb{R}^n$, $SO(n)$

standing assumptions:

$G \leq SL_d(\mathbb{R})$ real algebraic gp $/\mathbb{Q}$, of class $\mathcal{H}$.

$\Gamma = G \cap SL_d(\mathbb{Z})$ lattice in G

$\forall \delta > 0$, $K(\delta) = \{ [g] \in G/\Gamma$ s.t. $g\mathbb{Z}^d \cap \overline{B_\delta(0)} = \{0\} \}$

is compact subset of G/Γ, $\bigcup_{\delta > 0} K(\delta) = G/\Gamma$.

Def: We say $\phi(R)$ is of iterated-log class with ℓ logarithms if $\phi(R) \geq c \underbrace{\log \log \cdots \log}_{\ell} R$.

Theorem 1 (Margulis, Mohammadi, Shah, L*) Let $u_t < G$ be a one parameter unipotent gp. Then there is a ϕ of iterated log class such that for any $R, T > 0$, $x_0 \in G/\Gamma$ at least one of the following holds:

(1) for any $y \in K(\frac{1}{\phi(R)})$, $\exists\, |t| < T$ s.t. $d(u_t . x_0, y) < \phi(R)^{-1}$.

(2) there is $L < G$ of class $\mathcal{H}$, $y_0 \in G/\Gamma$ s.t. $\mathrm{vol}(L.y_0) < R$ so that for any $|t| < T$, there is a $\|g\| < R$ with

$$\max_{|s| < \frac{T}{R}} d(u_{s+t} . x_0, gL.y_0) < \frac{1}{R}$$

* in preparation

the # of logarithms in thm 1: $2(r+u-1)$

where $r = \mathrm{rk}_{\mathbb{R}} G$, $u = \max(\dim U : U < G \text{ unip.})$

Applying Thm 1 with fixed x_0 and first $T \to \infty$, then $R \to \infty$ we recover the Orbit Closure rigidity thm (for G algebraic, Γ arithmetic).

Key to proof: robust notion of Diophantine genericity of $x \in G/\Gamma$ with respect to $\{u_t : |t| < T\}$ (following Dani-Margulis)

Strategy of pf of orbit closure rigidity via

<u>measure classification</u> (Ratner)

Let $x_0 \in G/\Gamma$ be given.

Take $\mu_T = \frac{1}{T}\int_0^T \delta_{u_t . x}\, dt$, μ w^*-lim pt of $\{\mu_T\}$

If G/Γ not compact: need to use nondivergece properties of u_t orbit (Margulis, Dani)

via <u>measure classification thm:</u> write $\mu = \sum \mu_i$

μ_i living on immersed submanifolds $\mathcal{H}_i \subset G/\Gamma$.

Use the fact that U_t is polynomial in t to show if $\mu(\mathcal{H}_{j_0}) > 0$ then $x_0 \in \mathcal{H}_{j_0}$.

Conclude $\overline{\{U_t \cdot x_0\}_{t \in \mathbb{R}}}$ is homogeneous.

Strategy of pf of orbit closure thm via
<u>Diophantine Genericity</u> (EMMV, related to DM)

Background:

Thm (Dani; Margulis): (1) given $\varepsilon>0$, $\delta>0$, $\exists\, \varepsilon_1<\varepsilon$ s.t. for all $T\geq 0$, $[g]\in K(\varepsilon)$

$$\frac{1}{2T}\left|\{|t|<T : u_t\cdot[g]\in K(\varepsilon_1)\}\right|\geq(1-\delta)$$

(2) given $\delta>0$, $\exists\, \varepsilon_0=\varepsilon_0(\delta)$ s.t. for all $[g]$ either

$$\frac{1}{2T}\left|\{|t|<T : u_t\cdot[g]\in K(\varepsilon_0)\}\right|\geq(1-\delta) \qquad \forall\, T>T([g])$$

or $\exists$ $\mathbb{Q}$-parabolic $P<G$ s.t. $g\in P\Gamma$

Dani–Margulis (Shah, ...) linearization method gives similar results for avoiding periodic orbits $L.[g_0]$ for $L \supseteq \{u_t\}$.

Starting pt: define notion of "ε – diophantine for $|t| \leq T$"

key features: • for studying orbit $\{u_t . x : |t| < T\}$

ε-diophantine for $|t| \leq T$ has same info as

ε-diophantine for $|t| \leq T_1$ for $T_1 > T$

• $\forall \delta\, \exists \varepsilon_0$ s.t. if x is ε – diophantine for $|t| < T$,

$$\left| \left\{ |\tau| < T : u_\tau . x \text{ is not } \varepsilon_0\text{-diophantine for } |t| < T^{\alpha(\varepsilon)} \right\} \right| \leq (1-\delta) T$$

Basic strategy (qualitatively): Given $U \subset G$ unipotent, non-horospheric, $Y \subset G/\Gamma$ U-inv, closed, $x_0 \in Y$ "ε_0-Diophantine generic", define for $y \in Y$

$$\varphi_{U,Y}(y) = \{n \in N(U)/U : n.y \subset Y\}.$$

Then $\varphi_{U,Y}(u.y) = \varphi_{U,Y}(y)$ $\forall u \in U$, $\varphi_{U,Y}$ semicont.

let $x_1 \in \overline{Ux_0} \cap (\varepsilon_0\text{-Diophantine generic})$

s.t. $\varphi_{U,Y}(x_1)$ maximal.

show $\exists U_1 \not\geq U$ s.t. $U_1 . x_1 \subset Y$.

Effective results on Oppenheim Conj

Thm (Margulis, c. 85): let Q be indef., irrational, ternary quadratic form. Then $\delta_Q(N) = \min(\{Q(z) : 0 < \|z\| \leq N) \xrightarrow[N\to\infty]{} 0$

let $Q_0(x,y,z) = 2xy - z^2$, write $Q = c\, Q_0 \circ g$

$[g] = g\, SL_3(\mathbb{Z}) \in SL_3(\mathbb{R}) / SL_3(\mathbb{Z})$

Q rational $\Longleftrightarrow$ $SO(Q_0)[g]$ periodic

$\delta_Q(N) \longrightarrow 0 \Longleftrightarrow$ $SO(Q_0)[g]$ unbounded

(note: periodic & unbounded not mutually exclusive for $SO(Q_0)$ orbits)

Assume $\det Q = 1$, $\|Q\| \ll 1$, $\delta(N) = \min(\{|Q(z)| : 0 < \|z\| < N\})$

Thm 1 implies $\delta(N) \ll (\log\log\log\log\log\log N)^{-1}$ unless Q very close to rational form of low height.

Using observation of Dani-Margulis, more efficient argument can do better:

Thm (Margulis-L): $\delta(N) \ll (\log N)^{-c}$ unless Q very close to rational forms of low height.

For one parameter family of forms, Bourgain has a much better (optimal) bound.

Thm (Bourgain, March 2016): Let $Q(x,y,z) = x^2 - \alpha y^2 - \beta z^2$. for α fixed, for a.e. β,

$$\delta_Q(N) \ll N^{-1+\varepsilon} \quad \text{(assuming Lindelöf)}$$

$$\delta_Q(N) \ll N^{-2/5+\varepsilon} \quad \text{(unconditional)}$$

Thank You!