Life in characteristic p

(and how to escape it)

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&

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Road map

Road map

Number Theory

Road map

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$\mathbf{Z}$

Road map

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$$\mathbf{Z} \longleftrightarrow \mathbf{F}_p[x]$$

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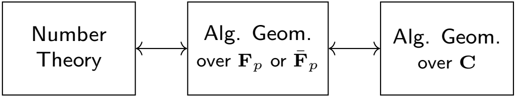

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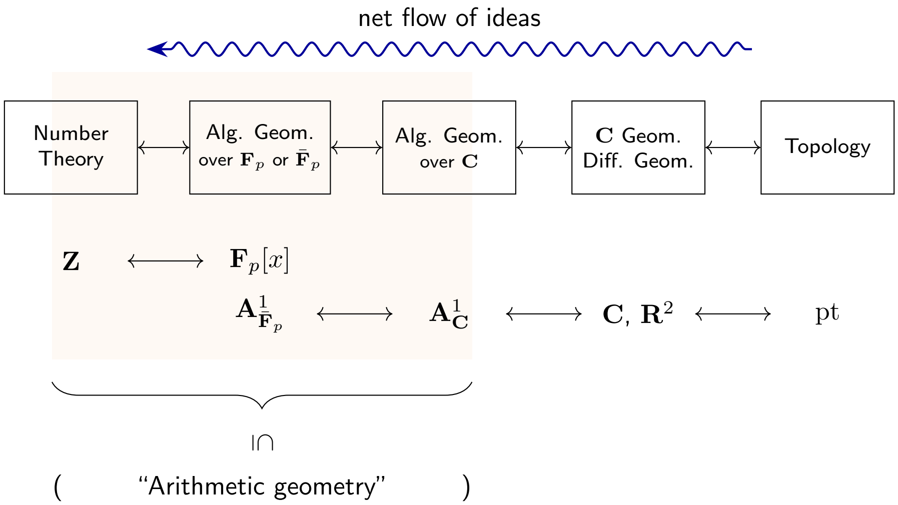
net flow of ideas
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This is never 0, so f is a covering space ("local diffeomorphism").

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Avoid finitely many bad primes: 2 and 17.

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This does not deserve to be called a lift.

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It is a quotient of a variety that lifts by an action that doesn't lift.

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This is analogous to Chow's lemma and resolution of singularities:

Question. Given *bad* X, does there exist *good* Y with $Y \twoheadrightarrow X$?

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Result

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Theorem (vDdB)

For any prime p, there exists a smooth projective surface X over $\bar{\mathbf{F}}_p$ such that no smooth projective variety Y admitting a surjection $Y \twoheadrightarrow X$ can be lifted to characteristic 0.

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For any prime p, there exists a smooth projective surface X over $\bar{\mathbf{F}}_p$ such that no smooth projective variety Y admitting a surjection $Y \twoheadrightarrow X$ can be lifted to characteristic 0.

It is a general divisor $X \subseteq C \times C \times C$ for any supersingular curve C with $g \geq 2$.

How to escape

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You can't!