Vertex algebras and quantum master equation

Si Li
(YMSC @ Tsinghua University)

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Motivation: Quantum B-model

$$\text{A-model (symplectic)} \xleftrightarrow{\textit{mirror}} \text{B-model (complex)}$$

$\boxed{\text{Gromov-Witten type theory}}$	$\boxed{\text{Hodge type theory}}$
counting genus zero curves	Variation of Hodge structures
counting higher genus curves	?

Question

What is the geometry of higher genus B-model? In other words, what is the quantization of VHS on CY geometry?

Motivation: Integrable hierarchy

Consider

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Promote u to be a field $u(x)$, introduce the Poisson bracket

$$\{u(x), u(y)\} = \partial_x \delta(x-y).$$

Then we find infinite number of pairwise commuting Hamiltonians

$$h_k = \frac{1}{(k+1)!} \oint dx u^{k+1}, \quad k \geq 1.$$

This is the dispersionless KdV integrable hierarchy.

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Question

Why integrable hierarchies in topological string? Is the exponential map $e^{u/t}$ universal?

We will be focused on the *B-twisted* topological string.

- [Bershadsky-Cecotti-Ooguri-Vafa, 1994]: B-model on CY three-fold can be described by a gauge theory

  $\rightarrow$ **Kodaira-Spencer gauge theory**.

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Topological B-model

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- [Costello-L, 2012]: The full description of Zwiebach's string field action in the B-model on arbitrary CY geometry

  $\rightarrow$ **BCOV theory**.

Motivation revisited

1. Higher genus B-model is described by the quantization of BCOV theory in the Batalin-Vilkovisky formalism.

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2. Let X be a CY geometry. Consider the fibration

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Start with B-model on $X \times \Sigma$, and compactify along X

$$\Longrightarrow \text{effective 2d chiral QFT on } \Sigma.$$

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- BV master equation on $\Sigma \Longrightarrow$ integrability.
- The leading effective action is computed by Saito's primitive form/Barannikov-Kontsevich's semi-infinite period map, which is the analogue of $e^{u/t}$.

Differential Batalin-Vilkovisky algebra

BV-formalism is a general method to quantize gauge theory.

Definition

A *differential BV algebra* is a triple $(\mathcal{A}, Q, \Delta)$ where

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$$\{a, b\} = \Delta(ab) - (\Delta a)b \mp a\Delta b, \quad \forall a, b \in \mathcal{A}.$$

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$$\{a, b\} = \Delta(ab) - (\Delta a)b \mp a\Delta b, \quad \forall a, b \in \mathcal{A}.$$

- Q and Δ are compatible: $Q\Delta + \Delta Q = 0$.

A toy model of differential BV structure

Let (V, Q, ω) be a (-1)-symplectic dg vector space

$$\omega \in \wedge^2 V^*, \quad Q(\omega) = 0, \quad \deg(\omega) = -1.$$

It identifies

$$V^* \simeq V[1].$$

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Let $K = \omega^{-1} \in \mathrm{Sym}^2(V)$ be the Poisson kernel under

$$\begin{array}{ccc} \wedge^2 V^* & \simeq & \mathrm{Sym}^2(V)[2] \\ \omega & & K \end{array}$$

Let $\mathcal{O}(V) := \widehat{\mathrm{Sym}}(V^*) = \prod_n \mathrm{Sym}^n(V^*)$. Then $(\mathcal{O}(V), Q)$ is a commutative dga.

A toy model of differential BV structure

The degree 1 Poisson kernel K defines a BV operator

$$\Delta_K : \mathcal{O}(V) \to \mathcal{O}(V) \quad \text{by}$$

$$\Delta_K(\varphi_1 \cdots \varphi_n) = \sum_{i,j} \pm (K, \varphi_i \otimes \varphi_j) \varphi_1 \cdots \hat{\varphi}_i \cdots \hat{\varphi}_j \cdots \varphi_n, \quad \varphi_i \in V^*.$$

Then $(\mathcal{O}(V), Q, \Delta_K)$ defines a differential BV algebra. We have

$$(-1)\text{-shifted dg symplectic} \Longrightarrow \text{differential BV}.$$

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Remark: this process is well-defined for Poisson instead of symplectic. In fact, as we will see, the Poisson kernel for topological B-model is degenerate.

BV-master equation

Let $(\mathcal{A}, Q, \Delta)$ be differential BV. Let $I = I_0 + I_1\hbar + \cdots \in \mathcal{A}[[\hbar]]$.

Definition

I is said to satisfy **quantum BV-master equation(QME)** if

$$\boxed{(Q + \hbar\Delta)e^{I/\hbar} = 0}.$$

This is equivalent to

$$QI + \hbar\Delta I + \frac{1}{2}\{I, I\} = 0.$$

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The leading $\hbar$-order I_0 satisfies

$$\boxed{QI_0 + \frac{1}{2}\{I_0, I_0\} = 0}$$

which is called the **classical BV-master equation(CME)**.

BV-master equation

Quantum master equation arises as the quantum consistency condition for quantum field theory with gauge symmetries. At the classical level, classical master equation says that

$$Q + \{I_0, -\}$$

squares zero, which describes the infinitesimal gauge transformations. In mathematical terminology, this defines an L_∞-algebra.

QFT case

QFT deals with infinite dimensional geometry. Typically the toy model (V, Q, ω) is modified to $(\mathcal{E}, Q, \omega)$ as follows:

V	$\mathcal{E} = \Gamma(X, E^\bullet)$
$Q : V \to V$	elliptic complex: $\cdots E^{-1} \stackrel{Q}{\to} E^0 \stackrel{Q}{\to} E^1 \cdots$
$\omega \in \wedge^2 V^*$	$\omega(s_1, s_2) = \int_X (s_1, s_2)$ where $(-,-) : E^\bullet \otimes E^\bullet \to \mathrm{Dens}_X$
V^*	$\mathcal{E}^*$: distributions on X
$(V^*)^{\otimes n}$	$(\mathcal{E}^*)^{\otimes n}$: distributions on X^n
$K_0 = \omega^{-1}$	K_0 is δ-function, which is a distribution on $X \times X$ supported on the diagonal

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The serious problem (UV-divergence) is that

$$\Delta_{K_0} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

is **ill-defined** since we can not pair two distributions. *Renormalization* is required!

Costello's homotopic renormalization

The basic idea is

$$H^*(distribution, Q) = H^*(smooth, Q).$$

Therefore we can replace K_0 by something smooth, and remember the original theory in a homotopic way.

Effective BV-formalism

Since $Q(K_0) = 0$, we can find P_r such that

$$K_0 = K_r + Q(P_r)$$

and K_r is *smooth*. Therefore $\Delta_{K_r} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$ is well-defined.

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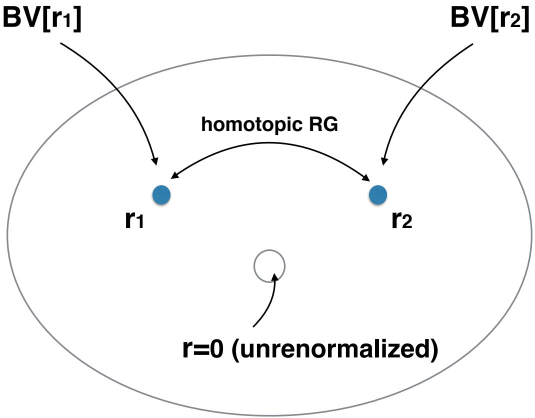
- BV master equation is formulated homotopically.

Effective BV quantization



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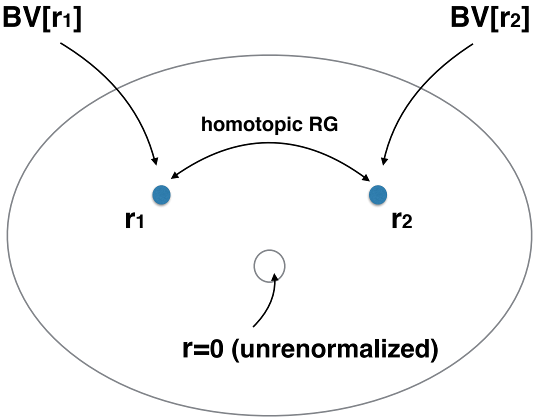

Definition

We say the theory is UV finite if $\lim_{r \to 0} \mathrm{BV}[r]$ exists.

2d Chiral QFT

Some examples of free CFT in 2d

- free boson: $\int \partial\phi \wedge \bar{\partial}\phi$.
- bc-system: $\int b \wedge \bar{\partial}c$.
- $\beta\gamma$-system: $\int \beta \wedge \bar{\partial}\gamma$.

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We will study effective BV quantization in 2d for chiral deformation of free CFT's of the form:

$$S = \text{free CFT's} + I.$$

Here

$$I = \int d^2 z \mathcal{L}^{hol}(\partial_z \phi, b, c, \beta, \gamma)$$

where $\mathcal{L}^{hol}$ is a lagragian density involving only holomorphic derivatives of $\partial_z\phi, b, c, \beta, \gamma$.

Vertex algebras

A *vertex algebra* is a vector space $\mathcal{V}$ with structures

- state-field correspondence

$$\mathcal{V} \to End(\mathcal{V})[[z, z^{-1}]]$$
$$A \to A(z) = \sum_n A_{(n)} z^{-n-1}$$

- vacuum: $|0\rangle \to 1$.
- translation operator, locality, etc.

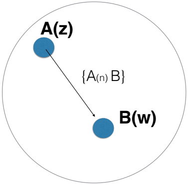
Operator expansion product (OPE)

We can define OPE's of fields by

$$A(z)B(w) = \sum_{n\in\mathbb{Z}} \frac{(A_{(n)} \cdot B)(w)}{(z-w)^{n+1}}$$

or simply the singular part

$$A(z)B(w) \sim \sum_{n\geq 0} \frac{(A_{(n)} \cdot B)(w)}{(z-w)^{n+1}}.$$



Lie algebra of Fourier modes

$$\boxed{\text{Vertex algebra } \mathcal{V} \Rightarrow \text{Lie algebra } \oint \mathcal{V}}.$$

As a vector space, the Lie algebra $\oint \mathcal{V}$ has a basis given by $A_{(k)}$'s

$$\oint \mathcal{V} := \mathrm{Span}_{\mathbb{C}} \left\{ \oint dz z^k A(z) := A_{(k)} \right\}_{A \in \mathcal{V}, k \in \mathbb{Z}}.$$

The Lie bracket is determined by the OPE

$$\left[A_{(m)}, B_{(n)}\right] = \sum_{j \geq 0} \binom{m}{j} \left(A_{(j)} B\right)_{m+n-j}.$$

Example

Let $\mathbf{h}$ be a graded vector space with $\deg = 0$ symplectic pairing

$$\langle -, - \rangle : \wedge^2 \mathbf{h} \to \mathbb{C}.$$

We obtain a vertex algebra structure on the free differential ring

$$\mathcal{V}[\mathbf{h}] = \mathbb{C}[\partial^k a^i], \quad \{a^i\} \text{ is a basis of } \mathbf{h}, k \geq 0.$$

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(or $\mathcal{V}[[\mathbf{h}]] = \mathbb{C}[[\partial^k a^i]]$). The OPE's are generated by

$$a(z)b(w) \sim \left(\frac{i\hbar}{\pi}\right) \frac{\langle a, b \rangle}{z - w}, \quad \forall, a, b \in \mathbf{h}.$$

$\Rightarrow \mathcal{V}[\mathbf{h}]$ is a combination of *bc* and $\beta\gamma$ systems.

2d Chiral QFT

We consider QFT on Σ where

$$\Sigma = (\mathbb{C}, z), \quad (\mathbb{C}^*, e^{2\pi iz}), \quad \text{or} \quad (E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau), z)$$

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3. (-1)-sympletic pairing

$$\omega(\varphi_1, \varphi_2) := \int dz \wedge \langle \varphi_1, \varphi_2 \rangle, \quad \varphi_i \in \mathcal{E}.$$

$\Rightarrow$ effective BV formalism

Chiral interaction

Let $\mathbf{h}^\vee$ be the linear dual of $\mathbf{h}$. There is a natural map

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$$I = \sum \oint \partial^{k_1} a_1 \cdots \partial^{k_n} a_n \in \mathcal{V}[[\mathbf{h}^\vee]], \quad \text{where } a_i \in \mathbf{h}^\vee,$$

it is mapped to

$$\hat{I}[\varphi] := \sum \int dz \partial_z^{k_1} a_1(\varphi) \cdots \partial_z^{k_n} a_n(\varphi)$$

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Lemma

The triple $\left(\oint \mathcal{V}[[\mathbf{h}^\vee]], \delta, [-,-]\right)$ *defines a dgLa.*

Main Theorem

Theorem (L)

We consider the following 2d chiral QFT

$$S = \textit{free CFT} + \hat{I}, \quad I \in \oint \mathcal{V}[\mathbf{h}^\vee][[\hbar]].$$

1. *The theory is UV-finite.*
2. *Solutions of (homotopic) effective BV master equations*

$$\Leftrightarrow \delta I + \frac{1}{2}\left(\frac{i\hbar}{\pi}\right)^{-1}[I, I] = 0, \quad I \in \oint \mathcal{V}[[\mathbf{h}^\vee]][[\hbar]].$$

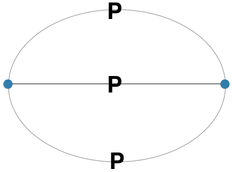
   In other words, I is a MC element of the dgLa $\oint \mathcal{V}[[\mathbf{h}^\vee]][[\hbar]]$.
3. *The generating functions are almost holomorphic modular forms, i.e., modular of the form* $\sum_{k=0}^{N} \frac{f_k(\tau)}{(\operatorname{im}\tau)^k}$.

Example: UV-finiteness and modularity

Consider the following chiral deformation of free boson on the elliptic curve E_τ

$$S = \frac{1}{2}\int_{E_\tau} \partial\phi \wedge \bar{\partial}\phi + \frac{1}{3!}\int_{E_\tau} \frac{d^2 z}{\operatorname{im}\tau}(\partial_z\phi)^3.$$

Let's look at a two-loop diagram



$$= \int_{E_\tau} \frac{d^2 z}{\operatorname{im}\tau} \, \mathbf{P}(\mathbf{z};\tau)^3.$$

Here the propagator is $\mathbf{P}(z;\tau) = \mathcal{P}(z,\tau) + \frac{\pi^2}{3}E_2^*$ where

$$\mathcal{P} \text{ Weierstrass P-function}, \quad E_2^* = E_2 - \frac{3}{\pi}\frac{1}{\operatorname{im}\tau}.$$

Naively $\mathcal{P}$ has a second order pole and $\int_{E_\tau} \mathcal{P}^3$ would be *divergent*.

However, in the sense of homotopic renormalization, its renormalized value has a well-defined limit $r \to 0$, whose value can be computed as follows:

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where $\mathfrak{M}$ is the sheaf of meromorphic functions, and Ω^{II} is the sheaf of abelian differentials of second kind. Then

$$\left[\mathbf{P}^3 \frac{dz}{\operatorname{im}\tau} \wedge d\bar{z}\right] \in H^1(E_\tau, \Omega^{\mathrm{II}}) \to H^2(E_\tau, \mathbb{C}) \xrightarrow{\int} \mathbb{C}.$$

represents the renormalized integral.

Geometric interpretation (joint in progress with Jie Zhou)

We find the following expression

$$\frac{1}{\pi^6}\int_{E_\tau}\frac{d^2 z}{\operatorname{im}\tau}\mathbf{P}^3 = \frac{2^2}{3^3 5}E_6 + \frac{2}{3^2 5}E_4 E_2^* - \frac{2}{3^3}(E_2^*)^3$$

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Under the $\bar{\tau}\to\infty$ limit, which amounts to replace $E_2^*\to E_2$,

$$\Rightarrow\frac{1}{\pi^6}\oint_A dz\mathcal{P}^3=\frac{2^2}{3^3 5}E_6+\frac{2}{3^2 5}E_4E_2-\frac{2}{3^3}(E_2)^3.$$

reducing to the A-cycle integral as computed by M.Douglas.

Let X be a CY. The field content of BCOV theory (in the generalized sense of [Costello-L]) is given by the complex

$$\mathcal{E} = \mathrm{PV}(X)[[t]], Q = \bar{\partial} + t\partial$$

where $\mathrm{PV}(X) = \Omega^{0,*}(X, \wedge^* T_X)$ and ∂ is the divergence operator w.r.t. the CY volume form.

BCOV theory on elliptic curve

We specialize to the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$

$$\begin{aligned}\mathcal{E} &= \Omega^{0,*}(E_\tau)[[t]] \oplus \Omega^{0,*}(E_\tau, T_{E_\tau}[1])[[t]] \\ &= \Omega^{0,*}(E_\tau) \otimes \mathbf{h}.\end{aligned}$$

where $\mathbf{h} = \mathbb{C}[[t, \theta]], \deg(t) = 0, \deg(\theta) = -1.$

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- $Q = \bar{\partial} + t\delta$, where $\delta = \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \theta}$.
- The Poisson kernel is degenerate. If we represent $\varphi \in \mathcal{E}$ by

$$\varphi = \sum_{k \geq 0} b_k t^k + \eta_k \theta t^k,$$

then the OPE's are generated by

$$b_0(z)b_0(w) \sim \frac{1}{(z-w)^2}, \quad \text{others} \sim 0.$$

b_0 is *dynamnical*, while $b_{>0}, \eta_\bullet$ are *background fields*.

Quantum B-model on elliptic curves

Theorem (L)

There exists a canonical solution of quantum master equation for BCOV theory on elliptic curves.

This is proved by analyzing the deformation obstruction complex of the dgLa for the relevant vertex algebra under the boson-fermion corresondence.

Stationary sector

Since the theory is UV-finite, we can express the solution of homotopic BV master equation via quantum corrected local functions. We give some explicit description in the so-called *stationary sector* (which amounts to freeze the background fields):

$$b_{>0} = 0, \quad \eta_{\bullet} = \text{constants}.$$

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The quantum corrected action in the stationary sector is

$$S = \int \partial\phi \wedge \bar{\partial}\phi + \sum_{k\geq 0} \int \eta_k \frac{W^{(k+2)}(b_0)}{k+2}, \quad b_0 = \partial_z \phi.$$

where

$$W^{(k)}(b_0) = \sum_{\sum_{i\geq 1} ik_i = k} \frac{k_i!}{\prod_i k_i!} \left(\prod_i \frac{1}{i!} (\sqrt{\hbar}\partial_z)^{i-1} b_0 \right)^{k_i} = b_0^k + O(\hbar).$$

are the bosonic realization of the $W_{1+\infty}$-algebra.

Quantization in stationary sector

In the stationary sector, the BV quantum master equation is equivalent to

$$\left[\oint W^{(k)}, \oint W^{(m)}\right] = 0, \quad \forall k, m \geq 0,$$

representing ∞ many commutating vertex operators.

Its classical limit is

$$\left\{\oint b_0^k, \oint b_0^m\right\} = 0, \quad \forall k, m \geq 0,$$

for the Poisson bracket $\{b_0(z), b_0(w)\} = \partial_z \delta(z - w)$ that we observe in the beginning.

Generating function

- The generating functions of the quantum BCOV theory are almost holomorphic modular forms. The $\bar{\tau}$-dependence is the famous holomorphic anomaly.
- In the stationary sector, the $\bar{\tau} \to \infty$ limit of the generating function can be computed by

$$\mathrm{Tr}\, q^{L_0 - \frac{1}{24}} e^{\frac{1}{\hbar} \sum_{k \geq 0} \oint_A \eta_k \frac{W^{(k+2)}}{k+2}}$$

which coincides with the stationary GW-invariants on the mirror elliptic curve computed by Okounkov-Pandharipande,

$$\Rightarrow \text{higher genus mirror symmetry.}$$

This generalizes the work of Dijkgraaf on the cubic interaction.

Remarks

- One way to understand the interaction of B-model on E is via

$$pt \times E \to E.$$

In general, we consider $X \times E \to E$, whose compactifiation along X gives rise to an effective 2d chiral theory on E. Then similarly we will find ∞ commutating Hamiltonians, which turns out to be Dubrovin-Zhang's *Principal integrable hierarchy* (in progress with Weiqiang He and Philsang Yoo).

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- Landau-Ginzburg twisting. Classical BCOV theory is equivalent to Saito's theory of primitive forms. The classical commuting hamiltonians can be computed by replacing $e^{u/t}$ with the so-called primitive form.
- Couple BCOV theory with Witten's HCS [Costello-L, 2016].

Thank You!