Multivariate trace inequalities

Marius Lemm

(IAS & Harvard)

IAS postdoc talks, October 2, 2017

What are trace inequalities?

Trace inequalities relate traces of various products of Hermitian matrices.

Typically, they become equalities for commuting matrices. Roughly speaking, they allow to control non-commutative terms "on average" (i.e., inside the trace).

Theorem (Golden and Thompson '65)

Let $A, B \in \mathbb{C}^{m \times m}$ be Hermitian matrices. Then

$$\mathrm{Tr}[e^{A+B}] \leq \mathrm{Tr}[e^{A}e^{B}]$$

- Compare this simple form to the elaborate expansion provided by the Baker-Campbell-Hausdorff formula.
- Lots of applications: Statistical mechanics; quantum information theory; random matrix theory.
- Follows from the Cauchy-Schwarz inequality.

Lieb's three-matrix inequality

It is not obvious how to generalize $\mathrm{Tr}[e^{A+B}] \le \mathrm{Tr}[e^A e^B]$ to more than two matrices. Naive generalizations are false:

$$\mathrm{Tr}[e^{A+B+C}] \not\le \begin{cases} \mathrm{Tr}[e^A e^B e^C], \\ \mathrm{Tr}[e^A e^{B/2} e^C e^{B/2}]. \end{cases}$$

But:

Theorem (Lieb '76)

Let $A, B, C \in \mathbb{C}^{m\times m}$ be Hermitian matrices. Then

$$\mathrm{Tr}[e^{A+B+C}] \le \int_0^\infty \mathrm{Tr}\left[e^A \frac{1}{e^{-B}+\tau I} e^C \frac{1}{e^{-B}+\tau I}\right] \mathrm{d}\tau$$

- The strange expression

  $$\int_0^\infty \frac{1}{X+\tau I} Y \frac{1}{X+\tau I} \mathrm{d}\tau = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \log(X+\epsilon Y)$$

  is a non-commutative analogue of $X^{-1}Y$.
- Proof uses convex matrix functions (Löwner's theorem helps).

A recent breakthrough

An extension of Lieb's three-matrix inequality to $n \geq 4$ matrices was missing, until last year.

Theorem (Sutter, Berta and Tomamichel 2016)

For $n \geq 2$, let $A_1, \ldots, A_n \in \mathbb{C}^{m\times m}$ be Hermitian matrices. Then

$$\mathrm{Tr}\left[\exp\left(\sum_{k=1}^{n} A_k\right))\right]$$
$$\leq \int_{-\infty}^{\infty} \mathrm{Tr}\left[e^{A_n} e^{\frac{1+it}{2}A_{n-1}} \ldots e^{\frac{1+it}{2}A_2} e^{A_1} e^{\frac{1-it}{2}A_2} \ldots e^{\frac{1-it}{2}A_{n-1}}\right] \beta(t)\mathrm{d}t$$

where $\beta(t) := \frac{\pi}{2}(1+\cosh(\pi t))^{-1}$ is an explicit probability density.

- For $n = 3$, this is actually Lieb's inequality in disguise.
- Their new $n = 4$ inequality is useful in quantum information theory.
- Proof uses complex interpolation (Stein-Hirschmann in Schatten spaces).

My two cents

Q: Can the SBT inequalities be formulated in Lieb's form (i.e., in terms of resolvents $\frac{1}{X+\tau I}$) for $n > 3$?

This is not just an academic question:
The unitaries $e^{\frac{it}{2}A_k}$ appear to block other applications of the new SBT inequalities, e.g., to random matrix theory (large deviations; bounds on the Lyapunov exponent). If the answer is yes, then we can use the "resolvent formalism" to remove the unitaries up to explicit commutators.

Theorem (arXiv:1708.04836)

A: Yes. E.g., for real symmetric matrices A_1, A_2, A_3, A_4*,*

$$\mathrm{Tr}_{\mathcal{H}}[e^{A_1+A_2+A_3+A_4}]$$
$$\leq \int_0^\infty \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}}\mathrm{Tr}\left[P\frac{1}{e^{-A_2}\otimes e^{-A_3}+\tau I}(e^{A_1}\otimes e^{A_4})\frac{1}{e^{-A_2}\otimes e^{-A_3}+\tau I}\right]\mathrm{d}\tau$$

Thank you for your attention!