Lifting Galois representations

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$X^\lambda(\mathbb{F}_p(a)) \neq \emptyset \iff \mathrm{St}^\lambda \cong \mathrm{St}^a$.

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The conjecture holds for $n = 2$ using the p-adic Langlands correspondence of Colmez.

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What if λ is not regular?