Lifting Galois representations

Daniel Le

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(Riemann) If X is an abelian variety of dimension g, then $H^1(X, \mathcal{O})$ and $H^0(X, \Omega^1)$ are g-dimensional and the Hodge filtration (with rational structure) determines X up to isogeny.

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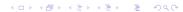
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Definition

A Serre weight is (an isomorphism class of) an irreducible $\overline{\mathbb{F}}_p$ -representation of $GL_n \mathbb{F}_p$.

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 $X^{\lambda}(\mathbb{F}_{p}(a)) \neq \emptyset \iff \operatorname{St}^{\lambda} \cong \operatorname{St}^{a}.$

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The conjecture holds for n = 1: If $\overline{\rho} = \overline{\mathbb{F}}_{\rho}(a)$, then $W(\overline{\rho}) = \mathrm{St}^{a}$.

The conjecture holds for n = 2 using the *p*-adic Langlands correspondence of Colmez.

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Theorem (L., Le Hung, Levin)

Let $\overline{\rho}$ be semisimple and generic (n is arbitrary). Then $\exists W^{?}(\overline{\rho})$ such that the conjecture (with $W^{?}(\overline{\rho})$ replacing $W(\overline{\rho})$) holds in the tamely potentially crystalline case when $\lambda = \eta$.

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What if λ is not regular?