# New corrections to mesoscopic level statistics for random band matrices

László Erdős
Institute for Science and Technology, Austria
Ludwig-Maximilians-Universität, Munich, Germany

Workshop on nonequilibrium dynamics and random matrices
IAS, Nov 4-8, 2013

Joint with Antti Knowles

# INTRODUCTION

**Universality conjecture for disordered quantum systems:**

A disordered quantum systems with sufficient complexity exhibits one of the following two behaviors:

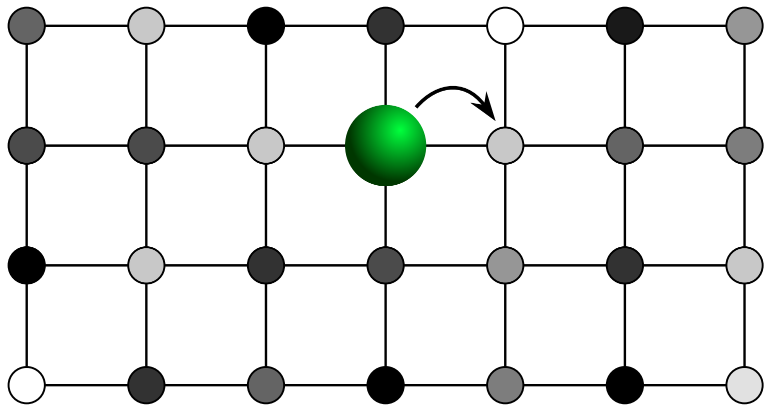
A) Localized evectors, lack of transport, and Poisson local spectral statistics (strong disorder)

B) Delocalization, quantum diffusion and random matrix (RMT) local statistics (weak disorder).

At first sight, localization is surprising (Anderson). Still, mathematically it is much more accessible (Fröhlich-Spencer, Aizenman-Molchanov, Minami, ...).

# Two popular models to study the dichotomy

(1) Random Schrödinger operators: in lattice box $\Lambda := [1, L]^d \cap \mathbb{Z}^d$



In $d = 1$ it corresponds to a narrow band matrix with i.i.d. diagonal:

$$H = -\Delta + \sum_x v_x = \begin{pmatrix} v_1 & 1 & & & & \\ 1 & v_2 & 1 & & & \\ & 1 & \ddots & & 1 & \\ & & & 1 & v_{L-1} & 1 \\ & & & & 1 & v_L \end{pmatrix}$$

Follows behavior (A) [Localization, Poisson]

(2) Wigner random matrices:

$$H = (H_{xy}), \qquad H = H^* \qquad \mathbb{E}H_{xy} = 0.$$

entries are identically distributed and independent up to symmetry.

$H$ models a mean-field hopping mechanism with random quantum transition rates. No spatial structure (dim is irrelevant or $d = 0$).

Follows behavior (B) [Delocalization, RMT]

Random band matrices: intermediate model that interpolates between (1) and (2).

They can be used to probe the transition between (A) and (B).

They also model quantum diffusion (today's focus)

# Random band matrices (RBM)

$\Lambda := [1, L]^d \cap \mathbb{Z}^d$ lattice box represents the configuration space.

$$H = (H_{xy})_{x,y\in\Lambda}, \qquad H = H^* \qquad \mathbb{E}H_{xy} = 0.$$

Entries are independent but no longer identically distributed. Variance is given by a band profile $f$ (even function, $\int_{\mathbb{R}^d} f = 1$)

$$s_{xy} := \mathbb{E}|H_{xy}|^2 = \frac{1}{W^d} f\Big(\frac{|x-y|}{W}\Big)$$

Key parameter: Band width $W \in [1, L]$ (range of interaction).

Nontrivial spatial structure like RS, but technically more accessible [Disertori, Pinson, Spencer, Zirnbauer, Shcherbina, Schenker...]

Normalization: Level spacing around energy $E$ is

$$\Delta = \frac{1}{L^d \varrho}, \qquad \varrho = \frac{1}{2\pi}\sqrt{4 - E^2}$$

# Linear statistics of eigenvalues in disordered systems

$$Y_\phi^\eta(E) := \sum_j \phi^\eta(\lambda_j - E), \qquad \text{with} \quad \phi^\eta(e) := \eta^{-1}\phi(e/\eta),$$

Eigenvalue density at energy $E$ on scale $\eta$ (smoothed by testfn $\phi$)

Question: Joint statistics of $Y_\phi^\eta(E_1), Y_\phi^\eta(E_2), \ldots$

Microscopic scale: $\eta \sim \Delta$: Poisson vs. RMT (GUE, GOE)

Macroscopic scale: $\eta \sim 1$: No universality, model dependent

Mesoscopic scale: $\Delta \ll \eta \ll 1$: Universalities with a phase transition.

Special physical motivation: fluctuation of conductance comes (partly) from the fluctuation of the number of states in a mesoscopic window around the Fermi level $E$ [Thouless]

The correlations of $Y_\phi^\eta(E_1), Y_\phi^\eta(E_2), \dots, Y_\phi^\eta(E_k)$ are equivalent to the truncated correlation functions smoothed on scale $\eta$, e.g.

$$\Big\langle Y_\phi^\eta(E-\frac{\omega}{2})\ ;\ Y_\phi^\eta(E+\frac{\omega}{2})\Big\rangle = \iint \phi^\eta(x-E+\frac{\omega}{2})\phi^\eta(y-E-\frac{\omega}{2})p^{(2)}(x,y)\mathrm{d}x\mathrm{d}y$$

E.g. for GUE, if the sine-kernel held on any scale, we had

$$\sim \int_{|e-\omega|\le\eta} \Big(\frac{\sin e/\Delta}{e/\Delta}\Big)^2 \mathrm{d}e \sim \frac{1}{\omega^2} \qquad \text{if} \quad \Delta \ll \eta \ll \omega \ll 1 \qquad (*)$$

Looks easy, by extrapolation: For GUE, (*) was indeed proved by Boutet de Monvel and Khorunzhy [1999]. However:

• Asymptotics $(*)$ does not hold for general delocalized systems: the sine-kernel fails on mesoscopic scale. Instead: Altshuler-Shklovskii formulas (1986 in physics, now we proved it at least for RBM).

• Sine kernel in $(*)$ may fail to predict the subleading term – New observation, contradicting to several physics predictions

Mesoscopic phase transition occurs at the Thouless energy

$$\eta_0 = \text{(time for diffusion to reach the boundary)}^{-1} = \frac{\text{diff coeff}}{L^2}$$

For RBM: $\eta_0 \sim W^2/L^2$ [E-Knowles, 2011]

## Altshuler-Shklovskii (AS) formulas

(1) In the diffusion regime, $\eta \gg \eta_0$

$$\text{Var}Y^\eta \sim (\eta/\eta_0)^{d/2} \qquad (d = 1, 2, 3)$$

$$\Big\langle Y_\phi^\eta(E - \frac{\omega}{2}) \; ; \; Y_\phi^\eta(E + \frac{\omega}{2}) \Big\rangle \sim \omega^{d/2-2} \qquad (d = 1, 3)$$

(2) In the mean field regime, $\eta \ll \eta_0$ , the same holds with $d = 0$.

- Compare with Poisson: Var $[\eta Y^\eta] \sim [\eta Y^\eta]$. Predicts the Anderson transition in $d = 1$ at $\ell \sim W^2$ via the crossover at $\eta \sim W^{-2}$.

- $d = 2$: critical case, leading term vanishes. Subleading terms?

**Theorem (Mesoscopic Universality for RBM)** [E-Knowles, 2013].

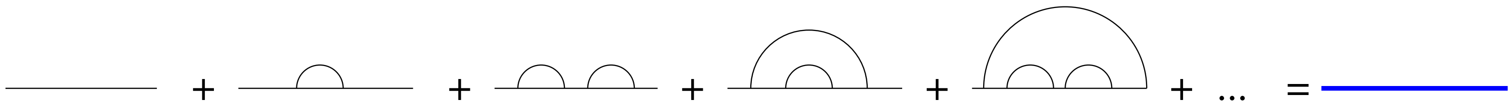
Suppose diffusive regime $\eta \gg \eta_0 = \left(\frac{W}{L}\right)^2$, and assume $\eta \gg W^{-d/3}$. Away from the spectral edges, we have for the density correlator

$$\frac{\left\langle Y^{\eta}_{\phi_1}(E_1)\,;\,Y^{\eta}_{\phi_2}(E_2)\right\rangle}{\left\langle Y^{\eta}_{\phi_1}(E_1)\right\rangle\left\langle Y^{\eta}_{\phi_2}(E_2)\right\rangle} = \frac{1}{(LW)^d}\,\Theta^{\eta}_{\phi_1,\phi_2}(E_1,E_2)\Big(1+O(W^{-\varepsilon})\Big),$$

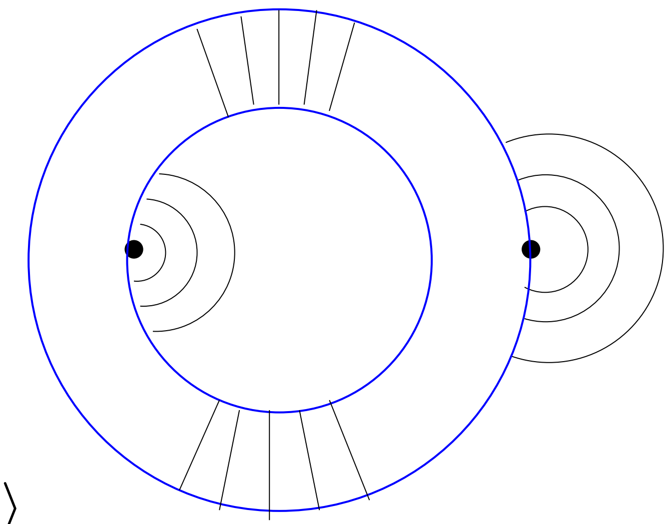
where $\Theta(E_1, E_2)$ is an explicit formula, depending only on the band profile $f$ but independent of the distribution of the matrix elements.

- The technical bound $\eta \gg W^{-d/3}$ is needed for the box band profile. For general $f$ we need $\eta \geqslant W^{-\varrho d}$ with some $\varrho > 0$.
- $\Theta =$ "one-loop diagrams after self-energy renormalization".
- Leading term is a higher order effect (cancellation in correlation)

## Computation of the leading term $\Theta$ (for $\omega \gg \eta$)



Self-energy renormalization of a single propagator



Leading term:
One-loop diagram
with two interparticle
and two intraparticle ladders

- $=$ traces in $\langle \mathrm{Tr}\,\mathrm{Im}G\ ;\ \mathrm{Tr}\,\mathrm{Im}G \rangle$

Interparticle ladders summed up as a geometric series:

$$\Theta \approx \frac{1}{\beta L^{2d}}\,\mathrm{ReTr}\frac{S}{(1-e^{i(E_1-E_2)}S)^2} * \phi_1^{\eta}(E_1) * \phi_2^{\eta}(E_2)$$

($\beta = 1, 2$ depending on the symmetry)

In F-space, with $\omega := E_2 - E_1$ and $\widehat{f}(q) \approx 1 - q \cdot Dq + O(q^4)$ for $q \approx 0$,

$$\frac{1}{L^{2d}} \operatorname{Re} \operatorname{Tr} \frac{S}{(1 - e^{i(E_1-E_2)}S)^2} \approx \frac{1}{(LW)^d} \operatorname{Re} \int_{\mathbb{R}^d} \frac{\widehat{f}(q)}{(1 - e^{i(E_1-E_2)}\widehat{f}(q))^2} \mathrm{d}q$$
$$\approx \frac{1}{(LW)^d} \operatorname{Re} \int_{\mathbb{R}^d} \frac{1 + O(q^2)}{(i\omega + q \cdot Dq)^2} \mathrm{d}q$$
$$\approx \frac{1}{(LW)^d} \frac{1}{\sqrt{\det D}} \cdot \omega^{d/2-2} \cdot \underbrace{\operatorname{Re} \int_{\mathbb{R}^d} \frac{\mathrm{d}x}{(i + x^2)^2}}_{=:K_d}$$

with $K_1 < 0$, $K_2 = 0$, $K_3 > 0$.

$d = 4$: Log-divergence

$d \geqslant 5$: Main part comes from $|q| \gtrsim 1$; it depends on the details of $f$.

$d = 2$: Expand $\widehat{f}$ up to fourth order and get

$$\Theta \sim \frac{Q - 1}{\beta(LW)^d} \Big( 0 \cdot \omega^{-1} + |\log \omega| \Big)$$

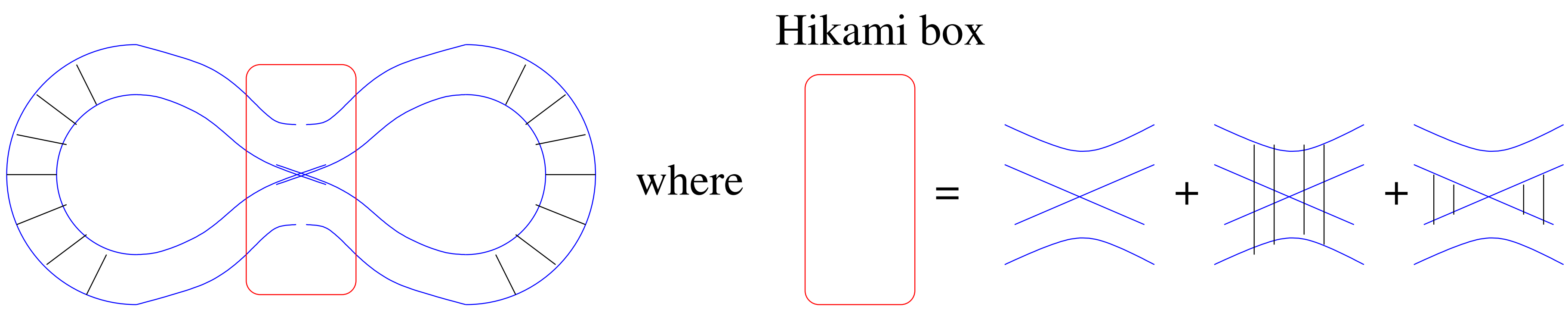
with a coefficient $Q := \int |D^{-1/2}x|^4 f(x) \mathrm{d}x$

## Weak localization correction in $d = 2$

Prediction [Kravtsov-Lerner, 1995]:

$$\Theta \sim \frac{1}{(LW)^d} \cdot \begin{cases} W^{-2}\omega^{-1} & \text{if } \beta = 1 \\ W^{-4}\omega^{-1} & \text{if } \beta = 2 \end{cases}$$

arises from the "two-loop" (figure-eight, or Hikami boxes) diagrams.




Main claim: There is a cancellation among these three diagrams.

Our rigorous result (comes from the first diag., no cancellation)

$$\Theta \sim \frac{1}{(LW)^d} \cdot |\log \omega|$$

# Mesoscopic correlations for Wigner matrices

Density-density correlator and its F.transform (called form factor)

$$R_E(s) := \frac{\left\langle Y^\eta_\phi(E - \frac{s}{2}\Delta)\, ;\, Y^\eta_\phi(E + \frac{s}{2}\Delta)\right\rangle}{\left\langle Y^\eta_\phi(E - \frac{s}{2}\Delta)\right\rangle \left\langle Y^\eta_\phi(E + \frac{s}{2}\Delta)\right\rangle}, \qquad K(\tau) := \int e^{-i\tau s} R(s) \mathrm{d}s$$

Wigner-Dyson statistics predicts (with Hikami correction )

$$R(s) = \frac{1}{(is)^2\beta} \cdot \begin{cases} 1 + \frac{1}{is} + \dots \\ 1 \end{cases} \qquad K(\tau) = \begin{cases} 2\tau - 2\tau^2 + \dots & \text{if } \beta = 1 \\ \tau & \text{if } \beta = 2 \end{cases}$$

Theorem [E-Knowles, 2013] For $L \times L$ (Bernoulli) Wigner matrices in the regime $\omega \gg \eta \geqslant L^{-1/2}$ we have

$$R(s) = \frac{1}{(is)^2\beta}\Big[1 + \frac{(L\eta)^2}{s^2} + \frac{s^2}{L^2} \dots + \frac{L}{s^2} \cdot \delta_{1,\beta} + \dots\Big] \qquad (s := L\omega \gg L^{1/2})$$

Blue: corrections to the usual one-loop diagram

Red: Uncancelled term from the Hikami box.

$$R(s) \overset{(WD)}{=} \frac{1}{(is)^2}\left[1 + \frac{1}{is} + \ldots\right], \qquad R(s) \overset{(Thm)}{=} \frac{1}{(is)^2}\left[1 + \frac{L}{s^2} + \ldots\right]$$

There are at least three "folkore" physics arguments for (WD):

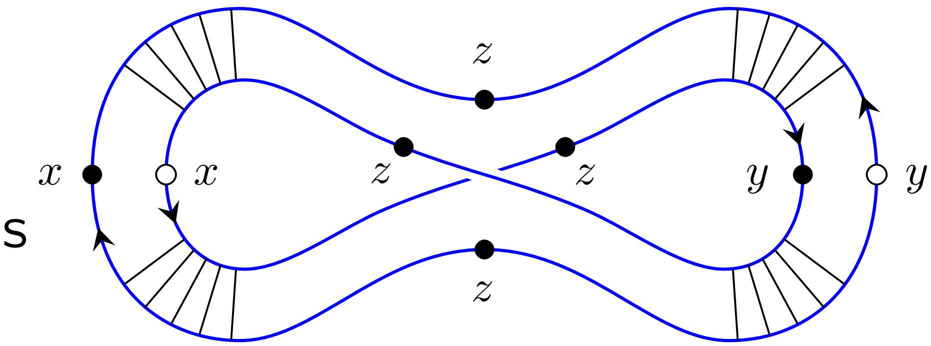
(1) Diagrammatic resolvent perturbation [Kravtsov, Lerner etc.]
(2) Semiclassical periodic orbit theory [Müller, Haake etc.]
(3) Sigma-model calculations [Efetov, Altland etc]

(1) and (2) are potentially unstable for $s \ll L$ (exponentially many diagrams/periodic orbits). But: they reproduce WD for $s \sim O(1)$ ["worst case"] $\Longrightarrow$ "no doubt" about their applicability through the whole mesoscopic range $1 \ll s \ll L$. Generalizations to higher order.

(1) and (2) rely on the same figure-eight diagram (Hikami box) albeit with a very different interpretation

(3) is exact for $s = 0$ and deteriorates as $s$ gets larger. Seems to break down once $s \gtrsim L^{1/2}$. [joint with A. Altland, in prep.]

Culprit: The first diagram in the Hikami box that is larger than the other two, so there is no cancellation for $s \gg L^{1/2}$.




Power counting:

Each interparticle ladder $= \frac{1}{L}(1 + e^{i\omega} + e^{2i\omega} + \ldots) \sim \frac{1}{iL\omega} = \frac{1}{is}$
Each interparticle ladder $= \frac{1}{L}$ (strong oscillation)

4 interparticle ladders: $(is)^{-4}$ and 3 different vertex labels: $L^3$
Total size $R(s) \sim L^{-2} \cdot L^3 (is)^{-4} \sim L/s^4$

Other diagrams in the Hikami term have an intraparticle ladder:

$$L^{-2}\left(L^3 \frac{1}{L} \frac{1}{s^4} + L^2 \frac{1}{L} \frac{1}{s^3} + L \frac{1}{L} \frac{1}{s^2}\right) \ll \frac{L}{s^4}$$

## Further results

- Joint law of the mesoscopic densities $(Y^{\eta}_{\phi_1}(E_1), \ldots Y^{\eta}_{\phi_k}(E_k))$ is asymptotically Gaussian with covariance $\Theta^{\eta}_{\phi_i,\phi_j}(E_i, E_j)$.

  For $E_i = E$, the covariance is the $\dot{H}^{\alpha}$ scalar product, $\alpha = \frac{1}{2} - \frac{d}{2}$

- Critical band matrix in $d = 1$: $S_{xy} \sim |x - y|^{-2}$ behaves as $d = 2$.

- At criticality, for the number of states $\mathcal{N}(I)$ in $I$, we prove

$$\text{Var}\, \mathcal{N}(I) \sim W^{-d}\, \mathbb{E}\mathcal{N}(I)$$

  - Asymptotic independence of $\mathcal{N}(I)$, $\mathcal{N}(I')$ if $I \cap I'$.
  - Coeff. $W^{-d}$ (compressibility) is predicted by [Chalker-Kravtsov-Lerner] and is in accordance with multifractality exponents.

- Generalized hermitian RBM with complex variances:

$$\mathbb{E}|H_{xy}|^2 = W^{-d} f(u), \qquad u = \frac{x-y}{W}$$

$$\mathbb{E}H_{xy}^2 = W^{-d} f(u)(1-h(u))e^{ig(u)}$$

where the real functions $f \geq 0$, $0 \leq h \leq 1$ are even and $g$ is odd.

Crossover from $\beta = 1$ ($g = h = 0$) to $\beta = 2$ is determined by

$$\sigma := \inf_q \int_{\mathbb{R}^d} (x \cdot q - g(x))^2 f(x) \mathrm{d}x + \int h(x) f(x) \mathrm{d}x$$

($\sigma = 0$ means trivial phase and $h = 0$, i.e. $\mathbb{E}|H_{xy}|^2 = \mathbb{E}H_{xy}^2$)

# Some ideas about the proof

$$Y_\phi^\eta(E) = \mathrm{Tr}\ \phi^\eta(H-E) = \mathrm{Tr}\ \int_0^\infty \widehat{\phi}(\eta t) e^{itE} e^{-itH}$$

Step 1. Chebyshev-Fourier expansion

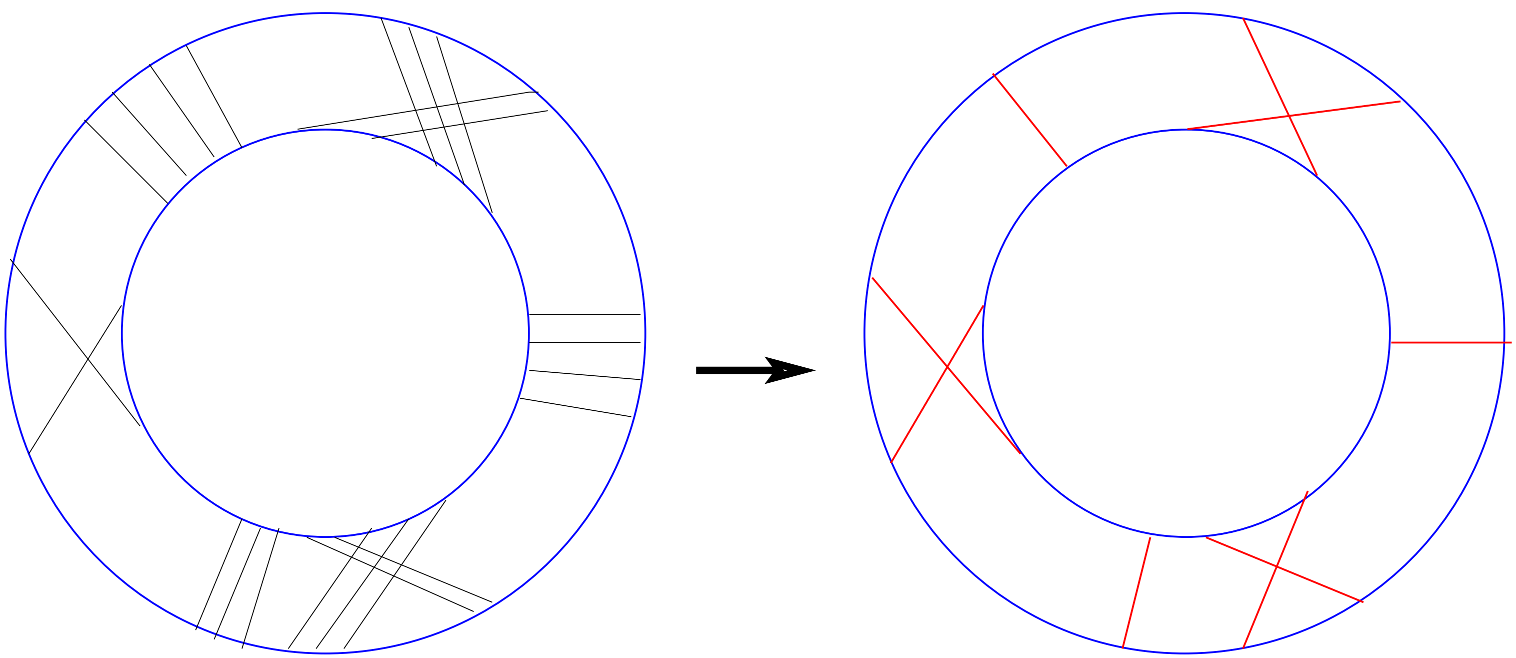
$$e^{-itH} = \sum_{n=0}^{\infty} a_n(t) H^{(n)}$$

in terms of non-backtracking powers [Feldheim-Sodin]

$$H^{(n)}_{x_0 x_n} = \sum_{x_j \neq x_{j+2}} H_{x_0 x_1} H_{x_1 x_2} \dots H_{x_{n-1} x_n}$$

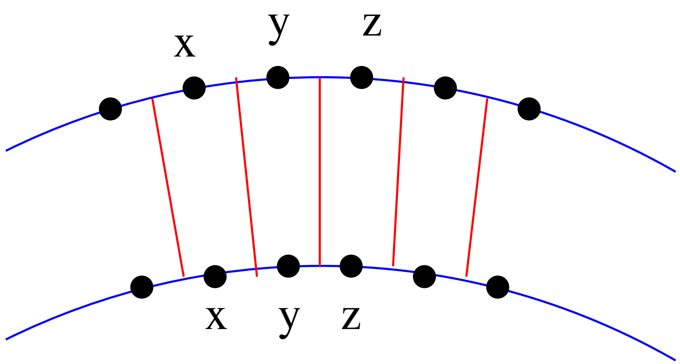
- More stable than Taylor, $\sum |a_n(t)|^2 = 1$
- Algebraic self-energy renormalization
- Exact only for $|H_{xy}| = 1$, estimates otherwise.

Step 2. Subladder resummation with oscillations, even in the regime where the sum is smaller than the summands. Random walk with phases. Unlike in our quantum diffusion work, here the phase cancellations need to be computed exactly since AS formula itself arises from a higher order term.

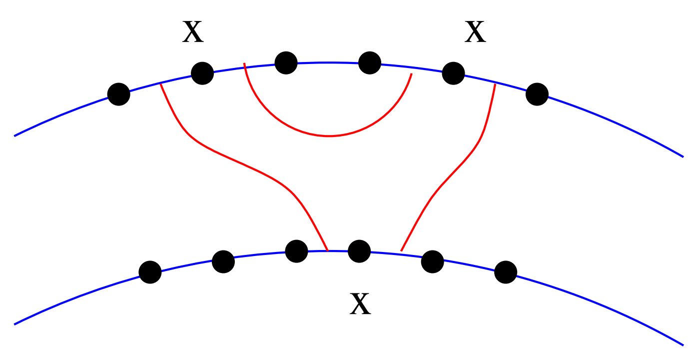


Resummation of ladders yields the skeleton of a graph

Step 3. Exponentially many error terms: classification of all graphs according to their size vs. complexity. Bound the number of labels in skeletons: 2/3 rule




Ladder: each label may occur only twice. Number of labels may be $n$ ($=$ no. of red lines)




Skeleton: each label occurs at least three times. There are at most $2n/3$ labels.

Step 4. Going from $\eta \gg W^{-d/3}$ to $\eta \gg W^{-d/2}$ (to reach the presumed optimal scale) requires to distinguish between inter/intraparticle ladders. This improves the previous 2/3-rule. Phases eliminate our previous "critical pairing" that saturated the 2/3-rule.

## SUMMARY

- Proof of the AS formulas for RBM: mesoscopic universality

- Rigorous weak localization corrections at criticality differ from previous physics predictions.

- New subleading term in the form factor for (Bernoulli) Wigner matrices on larger scales $\eta \gg L^{-1/2}$ ($s \gg L^{1/2}$).

- Physics predictions based upon Hikami boxes (diagrammatic or semiclassical orbit theory) are incorrect for larger scales. It seems more accurate for small scales despite that many more diagrams are neglected. How come?
  Threshold $\eta \ll L^{-1/2}$ for $\sigma$-model calculations.

- Open problem: similar analysis for random Schrödinger?