Mandelbrot Cascades and their uses

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Random multifractal measures

Mandelbrot Cascades are a class of **random** measures on $\mathbb{R}^n$ with non-trivial **multifractal** properties.

Cascade measure is a Borel measure $\mu(dx) = \mu(dx;\omega)$ on $x \in \mathbb{R}^n$, depending on $\omega \in \Omega$, a probability space.

μ has nontrivial **scaling properties**: for a ball B_r

$$\mathbb{E}\,\mu(B_r)^p \sim r^{\alpha(p)} \quad \text{as} \quad r \to 0$$

with $\alpha(p)$ a quadratic polynomial.

Gibbs measures

A one parameter family of cascade measures is given by

$$\text{"}\mu_\beta(dx) = e^{-\beta\phi(x)}dx\text{"}$$

$\beta \in [0, \infty)$ "inverse temperature".

$\phi(x) = \phi(x, \omega)$ is a **logarithmically correlated** random field

$$\mathbb{E}\,\phi(x)\phi(x') \sim \log\frac{1}{|x - x'|} \quad \text{as } |x - x'| \to 0$$

Proper definition requires a limiting process: μ_β is **not** continuous w.r.t. Lebesgue measure.

In 2d $\phi(x)$ is (a version) of the **Gaussian Free Field**, in 1d the $1/f$ **noise**.

Phase transition

These measures exhibit a **phase transition**: $\exists \beta_c$ s.t.

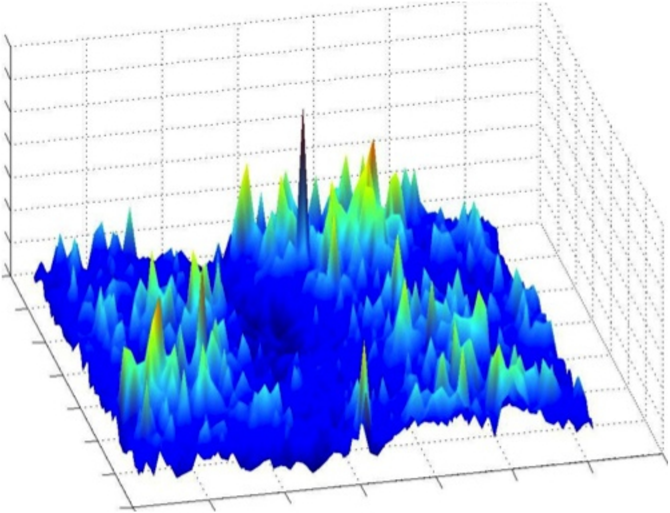
- For $\beta \leq \beta_c$, μ_β is continuous, singular w.r.t. Lebesgue
- For $\beta > \beta_c$, μ_β is atomic

They provide simple models of **freezing transition** believed to occur in (spin) **glasses**.

They have also been used to shed light on

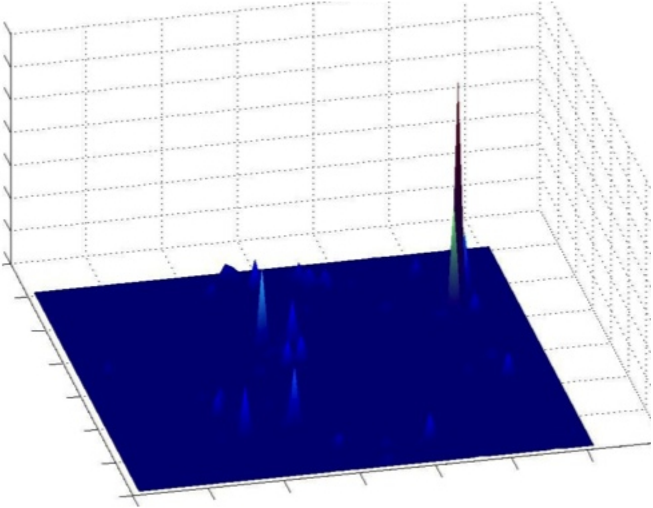
- **The KPZ relation** between **dimensions of fractals** in Euclidean and random geometry or more conjecturally **critical exponents** on regular and random surfaces (Duplantier and Sheffield)
- Random fractal plane curves via **conformal welding** (Astala, Jones, A.K. and Saksman; Sheffield)

$\beta < \beta_c$



Rhodes and Vargas (2013)

$\beta > \beta_c$



Rhodes and Vargas (2013)

Log correlated fields

Def. Logarithmically correlated random field ϕ in $\mathbb{R}^d$:

$$\mathbb{E}\phi(x)\phi(x') = \log|x - x'|^{-1} + g(x, y)$$

with g continuous.

- 2d free field with covariance $(-\Delta + 1)^{-1}$
- $1/f$ noise $x \in [0, 1]$, α_n, β_n i.i.d. $N(0, 1)$:

$$\phi(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(\alpha_n \cos 2\pi nx + \beta_n \sin 2\pi nx)$$

$$\mathbb{E}\phi(x)\phi(x') = \log|z - z'|^{-1}, \quad z = e^{2\pi ix}$$

Decomposition to scales

Log correlated fields may be decomposed in scales

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n(x)$$

- ϕ_n independent, fluctuations at scale 2^{-n}

$$\mathbb{E}\, \phi_n(x)\phi_n(x') = g_n(2^n x, 2^n x')$$

- $g_n(x, y)$ smooth, fast decay in $|x - y|$

Define also a regularized field with short distance cutoff 2^{-N}

$$\phi_{\leq N}(x) := \sum_{n=0}^{N} \phi_n(x)$$

Hierarchical field

Let $\mathcal{D}$ be the set of dyadic intervals $I \subset [0,1]$

Let $\{V_I\}_{I\in\mathcal{D}}$ be i.i.d.$\sim N(0,1)$ and set

$$\phi(x) = \sum_{I \ni x} V_I = \sum_{n=0}^{\infty} \phi_n(x)$$

where $\phi_n(x) = V_I$ for the unique I s.t. $|I| = 2^{-n}$ and $x \in I$. Then

$$\mathbb{E}\phi(x)\phi(x') = \sum_{I \ni x, x'} 1 = 1 + \log_2 d(x,x')^{-1}$$

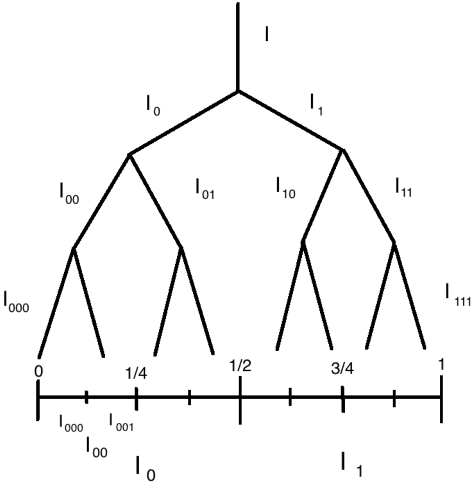
$d(x,x')$ length of shortest dyadic interval $I \ni x, x'$

Binary trees

Dyadic intervals in $[0,1] \leftrightarrow$ **binary trees** $\Sigma = \cup_{N=0}^{\infty} \Sigma_N$

$\Sigma_N = \{0,1\}^N$ lists edges (ancestors) of level N

$\sigma = \sigma_0 \sigma_1 \ldots \sigma_{n-1} \leftrightarrow$ interval $|I_\sigma| = 2^{-n}$



Directed polymer and branching random walk

• On each edge σ of the tree random weights V_σ

• The cutoff 2^{-N} field $\phi_{\leq N}(x)$ is constant $\equiv \phi_{\leq N}(\sigma)$ on the interval corresponding to $\sigma = \sigma_0\sigma_1 \ldots \sigma_N$:

$$\phi_{\leq N}(\sigma) = V_{\sigma_0} + V_{\sigma_0\sigma_1} + \cdots + V_{\sigma_0\sigma_1\ldots\sigma_N}$$

• Think of $\phi_{\leq N}(\sigma)$ as the **energy** of the **directed polymer** i.e. a path on the tree of length N from the root to σ

• We can also think of $\phi_{\leq N}(\sigma)$ as a **branching random walk**: at time N there are 2^N particles σ at positions $\phi_{\leq N}(\sigma)$

Multiplicative chaos

Let

$$\mu_{\beta,N}(dx) := e^{-\beta\phi_{\leq N}(x)}dx$$

Kahane's **multiplicative chaos** is the random measure

$$\nu_\beta = \lim_{N\to\infty} z_N \mu_{\beta,N}$$

whenever the limit exists for a (deterministic) constant z_N.

Density of $\mu_{\beta,N}$ is a **product** of independent random variables

$$e^{-\beta\phi_{\leq N}(x)} = \prod_{n=0}^{N} e^{-\beta\phi_n(x)}$$

Mandelbrot cascade

For hierarchical field this measure is the **Mandelbrot cascade** (1973)

It is the **Gibbs measure** of the directed polymer (Derrida-Spohn 1986):

$$\mathbb{P}(\text{path}) \propto e^{-\beta\phi_{\leq N}(\sigma)}$$

Martingale

Let $\mathcal{F}_N$ be the σ-algebra generated by $\{\phi_n\}_{n\leq N}$.

Since $\phi_{\leq N}(x) = \phi_N(x) + \phi_{\leq N-1}(x)$

$$\mathbb{E}(e^{-\beta\phi_{\leq N}(x)} \mid \mathcal{F}_{N-1}) = (\mathbb{E}e^{-\beta\phi_N(x)})e^{-\beta\phi_{\leq N-1}(x)}$$

Normalizing the measure as

$$\nu_{\beta,N} := \frac{e^{-\beta\phi_{\leq N}(x)}}{\mathbb{E}e^{-\beta\phi_{\leq N}(x)}}dx$$

(i.e. Wick ordering) we obtain

$$\mathbb{E}(\nu_{\beta,N} \mid \mathcal{F}_{N-1}) = \nu_{\beta,N-1}.$$

In particular total mass

$$M_N := \nu_{\beta,N}([0,1])$$

is a **martingale**:

$$\mathbb{E}(M_N \mid \mathcal{F}_{N-1}) = M_{N-1}, \;\; \mathbb{E}M_N = 1.$$

Uniform Integrability

M_N is a positive martingale $\implies$ it converges a.s. to $M \geq 0$.

To show $M > 0$ need uniform integrability, e.g. that $\mathbb{E}M_N^p$ stays bounded for some $p > 1$.

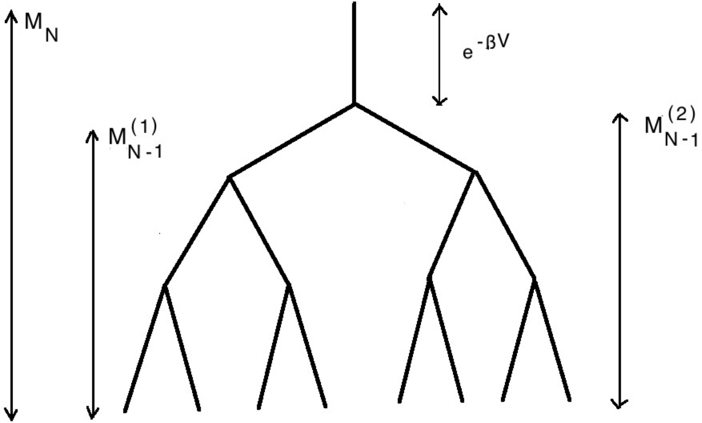
Kahane (1985) showed there exists a **critical** value β_c so that M_N is bounded in L^p for some $p > 1$ if and only if $\beta < \beta_c$.

In **hierarchical model** this is very easy to see using the tree structure.

Hierarchical Recursion relation

$$M_N \cong \frac{1}{2} e^{-\beta V - \frac{1}{2}\beta^2} (M_{N-1}^{(1)} + M_{N-1}^{(2)})$$

with $V \cong N(0,1)$, $M_{N-1}^{(i)}$ independent.



Uniform Integrability

$$M_N \cong \frac{1}{2} e^{-\beta V - \frac{1}{2}\beta^2} (M_{N-1}^{(1)} + M_{N-1}^{(2)})$$

Let $p > 1$. Using $(a+b)^p \geq a^p + b^p$ get

$$\mathbb{E} M_N^p \geq (\tfrac{1}{2})^p e^{\frac{1}{2}(p^2-p)\beta^2} 2\mathbb{E}\, M_{N-1}^p$$

Thus, if M_N converges in L^p then necessarily

$$(\tfrac{1}{2})^{p-1} e^{\frac{1}{2}(p^2-p)\beta^2} \leq 1 \quad \text{i.e.} \quad \beta^2 \leq (2\log 2)/p$$

and so, if $\beta \geq \sqrt{2\log 2}$, M_N can not converge in any L^p , p>1.

Converse is not much harder.

Also, the argument extends to Kahane's log correlated chaos.

Phase transition

Kahane: $\exists \beta_c$

- $M > 0$ almost surely for $\beta < \beta_c$
- $M = 0$ almost surely for $\beta \geq \beta_c$

Moreover $\lim_{N\to\infty} \nu_{\beta,N} = \nu_\beta$ almost surely and

- $\nu_\beta \neq 0$, (singular) continuous for $\beta < \beta_c$
- $\nu_\beta = 0$ for $\beta \geq \beta_c$
- We have also $M \in L^p(\Omega)$ for $p < (\beta_c/\beta)^2$ and

$$\mathbb{E}\nu(I)^p \sim C|I|^{\phi(p)}$$

  with $\phi(p) = p - (\frac{\beta}{\beta_c})^2(p - p^2)$

Is it possible to obtain a nontrivial measure ν_β for $\beta \geq \beta_c$?

Is it continuous? Atomic?

Liouville model

Find z_N s.t. the random variable

$$z_N \int_0^1 e^{-\beta\phi_{\leq N}(x)} dx$$

converges or, equivalently that its Laplace transform, i.e. the partition function of the "Liouville model"

$$F(\lambda, N) := \mathbb{E}\, e^{-\lambda z_N \int_0^1 e^{-\beta\phi_{\leq N}(x)} dx}$$

converges for all $\lambda \geq 0$ and is nontrivial.

We saw that for $\beta < \beta_c$ Wick ordering

$$z_N = 1/\mathbb{E}e^{-\beta\phi_{\leq N}(x)} = e^{-\frac{1}{2}\beta^2 \log 2^N} = e^{-\frac{\log 2}{2}\beta^2 N}$$

works. (also, Hoegh-Krohn (1971): $\beta < \beta_c/\sqrt{2}$)

Hierarchical Recursion relation

Consider the total mass of $2^N e^{-\beta\phi_{\leq N}(x)} dx$ i.e. the partition function of the directed polymer

$$Z_N = \sum_{\sigma \in \Sigma_N} e^{-\beta\phi_{\leq N}(\sigma)}.$$

It satisfies the recursion

$$Z_N \stackrel{d}{=} e^{-\beta V}(Z_{N-1}^{(1)} + Z_{N-1}^{(2)}),$$

Look at Laplace transform of Z_N in the variable $\lambda = e^{-\beta y}$, $y \in (-\infty, \infty)$:

$$G_N(y) := \mathbb{E} e^{-e^{-\beta y} Z_N}$$

For $\beta < \beta_c$ we saw $2^{-N} e^{-\beta^2 N} Z_N$ converges i.e.

$$G_N(y + c_\beta N)$$

has a limit as $N \to \infty$ if $c_\beta = \frac{1}{2}\beta + \log 2/\beta$.

Recursion relation

$$G_N(y) := \mathbb{E}e^{-e^{-\beta y}Z_N}, \quad Z_N \stackrel{d}{=} e^{-\beta V}(Z_{N-1}^{(1)} + Z_{N-1}^{(2)})$$

imply

$$\begin{aligned} G_{N+1}(y) &= \mathbb{E}(\exp(-e^{-\beta(y+V)}(Z_N^{(1)} + Z_N^{(2)}))) \\ &= \int \rho(v)E(\exp(-e^{-\beta(y+v)}Z_N))^2 dv \\ &= \int \rho(v)G_N(y+v)^2 dv. \end{aligned}$$

$\rho(v)$ density of V. Continuum limit $N \to \infty$ by **iteration**, initial data

$$G_0(y) = \exp(-e^{-\beta y}) \to \begin{cases} 0 & \text{if } y \to -\infty \\ 1 & \text{if } y \to \infty \end{cases}$$

Traveling wave

$$G_{N+1}(y) = \int \rho(v) G_N(y+v)^2 dv.$$

- $G_N \equiv 0$ is a linearly stable solution
- $G_N \equiv 1$ is a linearly unstable solution
- $G_N(y) = w_c(y - cN)$ traveling wave solutions

Given an initial datum G_0, G_N tends to a traveling wave with speed $c(\beta)$ **selected** by asymptotics of of G_0 at ∞ i.e. by the parameter β.

Indeed, our recursion is finite time version of the Fischer-Kolmogorov PDE.

Asymptotics

Recall

$$G_N(x) = \mathbb{E}\ \exp(-e^{-\beta y} Z_N)$$

Following Bramson's analysis for the PDE (1983) one gets

Theorem (C.Webb 2011)

$$\lim_{N\to\infty} G_N(y + m_{\beta,N}) \to g(y)$$

with ($\beta_c = \sqrt{2\log 2}$)

$$m_{\beta,N} = \begin{cases} \beta N & \text{if } \beta < \beta_c \\ \beta_c N - \frac{1}{2\beta_c}\log N & \text{if } \beta = \beta_c \\ \beta_c N - \frac{3}{2\beta_c}\log N & \text{if } \beta > \beta_c \end{cases}$$

Freezing: $g(y)$ is **independent** of β for $\beta \geq \beta_c$

Transition from **typical** to **extreme** configurations dominating the measure.

Convergence of the total mass

This gives the desired normalization since as $N \to \infty$:

$$\mathbb{E}\ \exp(-e^{-\beta(y+m_{\beta,N})} Z_N) = G_N(y + m_{\beta,N}) \to g(y)$$

Hence

$$e^{-\beta m_{\beta,N}} Z_N \to z_\beta \ \text{ as } \ N \to \infty$$

in distribution. In particular at the critical point this becomes

$$N^{\frac{1}{2}} \int_0^1 \frac{e^{-\beta \phi_{\leq N}(x)}}{\mathbb{E} e^{-\beta \phi_{\leq N}(x)}} dx \to z_{\beta_c}$$

i.e. the martingale is renormalized by $N^{\frac{1}{2}}$.

Similar results by Aïdekon & Shi, Madaule (2012).

Critical point and low temperature

Consequences of $N^{\frac{1}{2}}$ (J. Barral, A.K, M. Nikula, E. Saksman, C. Webb):

- ν_{β_c} is a.s. continuous, Hausdorff dimension zero
- Logarithmic modulus of continuity: for $\gamma < \frac{1}{2}$, almost surely

$$\nu_{\beta_c}(I) \leq C(\omega)|\log|I||^{-\gamma}$$

Consequence of freezing (Barral, Rhodes, Vargas):

- ν_β purely atomic for $\beta > \beta_c$.

Law of the low temperature measures

Recall the (renormalized) total mass has the law

$$\mathbb{E}\exp(-e^{-\beta y} z_\beta) = g(y) \ \ \forall \beta \geq \beta_c.$$

Put $t = e^{-\beta y}$. Then $t^{\frac{\beta_c}{\beta}} = e^{-\beta_c y}$ and thus

$$\mathbb{E}\exp(-t z_\beta) = \mathbb{E}\exp(-t^{\frac{\beta_c}{\beta}} z_{\beta_c}) \ \ \forall \beta \geq \beta_c.$$

Let for $\alpha \in (0,1)$ $L_\alpha(s)$, $s \geq 0$, be the stable Lévy process

$$\mathbb{E} e^{-tL_\alpha(s)} = e^{-st^\alpha}.$$

independent on z_{β_c}. Then

$$z_\beta \stackrel{d}{=} L_{\frac{\beta_c}{\beta}}(z_{\beta_c})$$

Law of the low temperature measures

Extends to measures (Barral, Rhodes and Vargas):

$$\nu_\beta([0,t]) \overset{d}{=} L_{\frac{\beta_c}{\beta}}(\nu_{\beta_c}([0,t])) \quad \text{for all } t \in [0,1]$$

$L_{\frac{\beta_c}{\beta}}$ pure jump process $\implies$ ν_β, $\beta > \beta_c$, **a.s. purely atomic**.

The critical measure determines the low temperature one.

Exponential of the Free field

In multiplicative chaos the renormalization group becomes non-local. However much can be done using convexity and the fact that the covariances of cascade and chaos are comparable.

• $\beta < \beta_c$: Martingale normalization gives nontrivial limit (Kahane, Bacry & Muzy)

• $\beta = \beta_c$: $N^{\frac{1}{2}} \times$ martingale normalization gives **continuous** measure (Duplantier, Rhodes, Sheffield, Vargas) of **zero dimension** (Barral, A.K., Nikula, Saksman,Webb).

• $\beta > \beta_c$: Normalization, freezing, atomicity (Madaule, Rhodes and Vargas)

• $\beta = \infty$ Let M_N be the maximum of $\phi_{\leq N}(x)$. Then $m_N - \mathbb{E}m_N$ converges in distribution (Bolthausen, Bramson, Zeitouni, Ding)

Random Geometry: KPZ

KPZ formula relates Hausdorff dimension of fractals in the Euclidean metric and their dimension in a random metric.

Define on $[0, 1]$ random metric

$$\rho_\beta(x, y) = \nu_\beta([x, y])$$

- Let $K \subset [0, 1]$
- ζ_0 Hausdorff dimension of K w.r.t. Euclidean metric
- ζ_β Hausdorff dimension of K w.r.t. random metric ρ_β

Then

- For $\beta \leq \beta_c$, $\zeta_0 = \zeta + (\frac{\beta}{\beta_c})^2 \zeta(1 - \zeta)$
- For $\beta > \beta_c$, $\zeta_\beta = \frac{\beta_c}{\beta} \zeta_{\beta_c}$

Duplantier and Sheffield, Benjamini and Schramm: $\beta < \beta_c$

Random Curves and Surfaces

Conformally invariant random plane curves

- Glueing discs with the random metric ν_β produces random curves, loop version of $\mathrm{SLE}_{\kappa(\beta)}$, $\kappa(\beta) < 4$ if $\beta < \beta_c$ (Astala, Jones, A.K. Saksman; Sheffield).
- How about $\beta \geq \beta_c$?

Random surfaces

- Riemannian metric $e^{-\beta\phi_N(z)}(dz)^2$ on a domain or S^2.
- Do we get as $N \to \infty$ a random metric space $\mathcal{M}_\beta$?
- What is the Hausdorff dimension of $\mathcal{M}_\beta$? ($\stackrel{?}{=}$ 4 for $\beta = 8/3$)?
- Is it a scaling limit of random triangulations weighted with Potts or $O(N)$ models?

Critical random band matrices

Critical random band matrices

$$\mathbb{E}|H_{ij}|^2 = (1 + |i-j|/b)^{-2}$$

Inverse participation rates

$$P_q = \sum_i |\psi_i|^q$$

expected to scale with volume as

$$P_q \propto N^{-\tau(q)}$$

with (localized) $0 < \tau(q) < q-1$ (extended). Let

$$H_{ij}(n) := H_{ij} 1_{|i-j| \le n}$$

For b small (small off-diagonal terms) Levitov derived a renormalization group equation for $P_q(n)$:

$$P_q(n+1) \stackrel{d}{=} \xi P_q^{(1)}(n) + (1-\xi) P_q^{(2)}(n)$$

ξ certain random variable on $[0,1]$.

Freezing transition

Mirlin and Evers used this to compute

$$\mathbb{E}P_q(N) \sim N^{-\tilde{\tau}(q)}$$

with

$$\tilde{\tau}(q) = \frac{2\Gamma(q-1)}{\sqrt{\pi}\Gamma(q-\frac{1}{2})}$$

Moreover, studying the tail of the pdf of P_q they concluded a transition at $q^* = 2.405..$:

$$\tau(q) = \tilde{\tau}(q), \;\; q \leq q^*, \;\; \tau(q) = \alpha q, \;\; q \geq q^*$$

Looks like a freezing transition as in cascade with logarithmic corrections at $q \geq q^*$ (Fyodorov).

Real challenge is to justify the RG!

Characteristic polynomial

Characteristic polynomial of $N \times N$ unitary matrix

$$p_N(x) := \det(1 - e^{-2\pi i x} U_N)$$

where $x \in [0, 1]$, Then

$$\log |p_N(x)| = -\frac{1}{2} \sum_{n=1}^{\infty} (e^{2\pi i n x} \operatorname{tr} U_N^n + e^{-2\pi i n x} \operatorname{tr} U_N^{-n})$$

Diaconis and Shahshahani: if U_N is CUE then for any M

$$\{ \sqrt{n} \operatorname{tr} U_N^n \}_{n \le M} \to \{ \frac{1}{\sqrt{2}} (a_n + i b_n) \}_{n \le M} \text{ as } N \to \infty$$

with a_n, b_n i.i.d. $N(0,1)$. Thus, formally

$$-\sqrt{2} \log |p_N(x)| \to \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (a_n \cos 2\pi n x + b_n \sin 2\pi n x)$$

the $1/f$ noise.

Characteristic polynomial

Does the limit

$$\lim_{N\to\infty} z_N |p_N(x)|^\beta dx = \nu_\beta(dx)$$

exist? Can it be realized as a martingale in N? (Bourgade, Hughes, Nikeghbali, Yor, showed for fixed x)

Does it exhibit a freezing transition?

Applications to ζ-function: (Fyodorov and Keating)

Conformal Welding

Conformal welding gives a correspondence between:

Closed curves in $\hat{\mathbb{C}}$ $\leftrightarrow$ **Homeomorphisms** $\phi : S^1 \to S^1$

Jordan curve $\Gamma \subset \hat{\mathbb{C}}$ splits plane $\hat{\mathbb{C}}$ to inside R and outside R^c.
Riemann mappings

$$f_+ : \mathbb{D} \to R \quad \text{and} \quad f_- : \mathbb{D}^c \to R^c$$

f_- and f_+ extend continuously to $S^1 = \partial\mathbb{D} = \partial\mathbb{D}^c \implies$

$$\phi = (f_+)^{-1} \circ f_- : S^1 \to S^1 \ \text{ Homeomorphism}$$

Welding problem: invert this:

Given $\phi : S^1 \to S^1$, **find** Γ and $f_\pm$.

Continuity

Continuity follows from

$$\nu_{\beta_c}(I_\sigma) \stackrel{d}{=} e^{-\beta_c \phi_{\leq N}(\sigma)} z_{\beta_c},$$

for $\sigma \in \Sigma_n$ and

$$n^{\frac{1}{2}} \sum_{\sigma \in \Sigma_n} e^{-\beta_c \phi_{\leq N}(\sigma)} \to z_{\beta_c}$$

and a tail estimate for z_{β_c}.

Proof of KPZ

For the upper bound need to control 1-point functions

$$\mathbb{E}\left(\rho_\beta(x,y)\right)^s \sim |x-y|^{\phi(s)}$$

where the multi-fractal exponent is explicit:

$$\phi(s) = s - (\frac{\beta}{\beta_c})^2(s - s^2)$$

For the lower bound need to estimate the 2-point function

$$\mathbb{E}\left(d\nu_\beta(x)d\nu_\beta(y)\right)$$

using hierarchical structure and scale invariance.

In low temperatures the Levy process induces a natural scaling.