Spectra of Large Random Stochastic Matrices & Relaxation in Complex Systems

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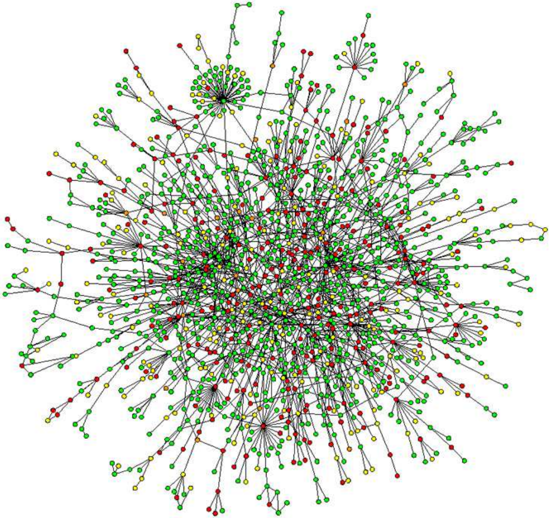
Aim of this talk

To study relaxation in complex systems using spectra of Markov transition matrices defined in terms of large random graphs.



Jeong et al, Nature (2001)

Outline

1. Introduction
   - Discrete Markov Chains
   - Spectral Properties – Relaxation Time Spectra
2. Relaxation in Complex Systems
   - Markov Matrices Defined in Terms of Random Graphs
   - Applications: Random Walks, Relaxation in Complex Energy Landscapes
3. Spectral Density
   - Replica Method – Unbiased Random Walk
   - Replica Method – General Markov Matrices
   - Analytically Tractable Limiting Cases
4. Numerical Tests
5. Summary

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Discrete Markov Chains

- Discrete homogeneous Markov chain in an N-dimensional state space,

$$\mathbf{p}(t+1) = W\mathbf{p}(t) \qquad \Leftrightarrow \qquad p_i(t+1) = \sum_j W_{ij} p_j(t) \ .$$

- Normalization of probabilities requires that W is a stochastic matrix,

$$W_{ij} \geq 0 \quad \text{for all } i,j \qquad \text{and} \qquad \sum_i W_{ij} = 1 \quad \text{for all } j \ .$$

- Implies that generally

$$\sigma(W) \subseteq \{z; |z| \leq 1\} \ .$$

- If W satisfies a detailed balance condition, then

$$\sigma(W) \subseteq [-1, 1] \ .$$

Spectral Properties – Relaxation Time Spectra

- Perron-Frobenius Theorems: exactly one eigenvalue $\lambda_1^\mu = +1$ for every irreducible component μ of phase space.
- Assuming absence of cycles, all other eigenvalues satisfy

$$|\lambda_\alpha^\mu| < 1 \,, \quad \alpha \neq 1 \,.$$

- If system is overall irreducible: equilibrium is unique and convergence to equilibrium is exponential in time, as long as N remains finite:

$$\mathbf{p}(t) = W^t \mathbf{p}(0) = \mathbf{p}_{\mathrm{eq}} + \sum_{\alpha(\neq 1)} \lambda_\alpha^t \, \mathbf{v}_\alpha \, \big(\mathbf{w}_\alpha, \mathbf{p}(0)\big)$$

- Identify relaxation times

$$\tau_\alpha = -\frac{1}{\ln|\lambda_\alpha|}$$

$\Longleftrightarrow$ spectrum of W relates to spectrum of relaxation times.

Outline

Markov matrices defined in terms of random graphs

- Interested in behaviour of Markov chains for large N, and transition matrices describing complex systems.
- Define in terms of weighted random graphs.
  - Start from a rate matrix $\Gamma = (\Gamma_{ij}) = (c_{ij} K_{ij})$
  - on a random graph specified by

$$\text{a connectivity matrix } C = (c_{ij}) \,, \quad \text{and edge weights } K_{ij} > 0 \,.$$

- Set Markov transition matrix elements to

$$W_{ij} = \begin{cases} \frac{\Gamma_{ij}}{\Gamma_j} & , \; i \neq j \,, \\ 1 & , \; i = j \,, \; \text{ and } \; \Gamma_j = 0 \,, \\ 0 & , \; \text{otherwise} \;\;, \end{cases}$$

where $\Gamma_j = \sum_i \Gamma_{ij}$.

Master-Equation Operator

- Master-equation operator related to Markov transition matrix W,

$$M_{ij} = \begin{cases} \frac{\Gamma_{ij}}{\Gamma_j} & , \ i \neq j \ , \\ -1 & , \ i = j \ , \text{and } \Gamma_j \neq 0 \ , \\ 0 & , \ \text{otherwise} \ , \end{cases}$$

in terms of which

$$p_i(t+1) - p_i(t) = \sum_j [W_{ij} p_j(t) - W_{ji} p_i(t)] = \sum_j M_{ij} p_j(t) \ .$$

- Special case: unbiased random walk, with $K_{ij} = 1$, so

$$W_{ij} = \frac{c_{ij}}{k_j} \ , \qquad k_j = \sum_i c_{ij}$$

for which

$$M_{ij} = \begin{cases} \frac{c_{ij}}{k_j} & , \ i \neq j \ , \\ -1 & , \ i = j \ , \text{and } k_j \neq 0 \\ 0 & , \ \text{otherwise} \ . \end{cases}$$

Symmetrization

- Markov transition matrix can be symmetrized by a similarity transformation, if it satisfies a detailed balance condition w.r.t. an equilibrium distribution $p_i = p_i^{\rm eq}$

$$W_{ij} p_j = W_{ji} p_i$$

- Symmetrized by $\mathcal{W} = P^{-1/2} W P^{1/2}$ with $P = \mathrm{diag}(p_i)$

$$\mathcal{W}_{ij} = \frac{1}{\sqrt{p_i}} W_{ij} \sqrt{p_j} = \mathcal{W}_{ji}$$

- Symmetric structure is inherited by transformed master-equation operator $\mathcal{M} = P^{-1/2} M P^{1/2}$,

$$\mathcal{M}_{ij} = \begin{cases} \mathcal{W}_{ij} & , \; i \neq j \;, \\ -1 & , \; i = j \;, \; \text{ and } k_j \neq 0 \\ 0 & , \; \text{otherwise} \;\; . \end{cases}$$

- **Computation of spectra below so far restricted to this case.**

Applications I – Unbiased Random Walk

- Unbiased random walks on complex networks: $K_{ij} = 1$; transitions to neighbouring vertices with equal probability:

$$W_{ij} = \frac{c_{ij}}{k_j} \ , \quad i \neq j \ ,$$

  and $W_{ii} = 1$ on isolated sites ($k_i = 0$).
- Symmetrized version is

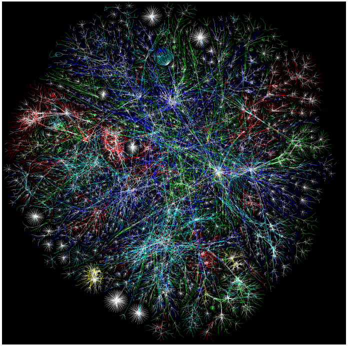
$$\mathcal{W}_{ij} = \frac{c_{ij}}{\sqrt{k_i k_j}} \ , \quad i \neq j \ ,$$

  and $\mathcal{W}_{ii} = 1$ on isolated sites.
- Symmetrized master-equation operator known as normalized graph Laplacian

$$\mathcal{L}_{ij} = \begin{cases} \frac{c_{ij}}{\sqrt{k_i k_j}} & , \ i \neq j \\ -1 & , \ i = j \ , \text{and } k_i \neq 0 \\ 0 & , \ \text{otherwise} \ . \end{cases}$$

Applicatons II – Non-uniform Edge Weights

- Internet traffic (hopping of data packages between routers)



Applicatons II – Non-uniform Edge Weights

- Internet traffic (hopping of data packages between routers)
- Relaxation in complex energy landscapes; Kramers transition rates for transitions between "inherent states"

$$\Gamma_{ij} = c_{ij} \exp\left\{ -\beta(V_{ij} - E_j) \right\}$$

with energies E_i and barriers V_{ij} from some random distribution.
$\Leftrightarrow$ generalized trap model.

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  with energies E_i and barriers V_{ij} from some random distribution.
  $\Leftrightarrow$ generalized trap model.
- Markov transition matrix of generalized trap model satisfies a detailed balance condition with

$$p_i = \frac{\Gamma_i}{\mathcal{Z}} \mathrm{e}^{-\beta E_i}$$

  $\Rightarrow$ can be symmetrized.

Outline

Spectral Density and Resolvent

- Spectral density from resolvent ($A = \mathcal{W}, \mathcal{L}, \mathcal{M}$)

$$\rho(\lambda) = \lim_{N\to\infty} \frac{1}{\pi N} \mathrm{Im}\,\mathrm{Tr}\left\langle \left[\lambda_\varepsilon \mathbb{1} - A\right]^{-1} \right\rangle, \qquad \lambda_\varepsilon = \lambda - \mathrm{i}\varepsilon$$

- Express as (S F Edwards & R C Jones, JPA, 1976)

$$\begin{aligned} \rho(\lambda) &= \lim_{N\to\infty} \frac{1}{\pi N}\, \mathrm{Im}\, \frac{\partial}{\partial\lambda}\, \mathrm{Tr}\left\langle \ln\left[\lambda_\varepsilon \mathbb{1} - A\right] \right\rangle \\ &= \lim_{N\to\infty} -\frac{2}{\pi N}\, \mathrm{Im}\, \frac{\partial}{\partial\lambda} \left\langle \ln Z_N \right\rangle, \end{aligned}$$

where Z_N is a Gaussian integral:

$$Z_N = \int \prod_k \frac{\mathrm{d}u_k}{\sqrt{2\pi/\mathrm{i}}} \exp\left\{ -\frac{\mathrm{i}}{2} \sum_{k,\ell} u_k (\lambda_\varepsilon \delta_{k\ell} - A_{k\ell}) u_\ell \right\}.$$

- Use Replica Method to perform average.

$$\left\langle \ln Z_N \right\rangle = \lim_{n\to 0} \frac{1}{n} \ln \left\langle Z_N^n \right\rangle$$

Replica – Unbiased Random Walk

- For unbiased random walk, decompose system into the set of non-isolated sites, $\mathcal{N} = \{i; k_i \neq 0\}$, and its complement $\bar{\mathcal{N}}$.
- Gives

$$Z_N = Z_{\bar{\mathcal{N}}} \times Z_{\mathcal{N}} \ , \quad \text{with } Z_{\bar{\mathcal{N}}} = (\lambda_\varepsilon - 1)^{-|\bar{\mathcal{N}}|/2} \ .$$

and

$$Z_{\mathcal{N}} = \int \prod_{i \in \mathcal{N}} \frac{\mathrm{d}u_i}{\sqrt{2\pi/\mathrm{i}}} \exp\left\{ -\frac{\mathrm{i}}{2} \sum_{i,j \in \mathcal{N}} \left(\lambda_\varepsilon \delta_{ij} - \frac{c_{ij}}{\sqrt{k_i k_j}} \right) u_i u_j \right\} \ ,$$

- Transform variables $u_i \leftarrow \frac{u_i}{\sqrt{k_i}}$ on $\mathcal{N}$, to get

$$Z_{\mathcal{N}} = \left(\prod_{i \in \mathcal{N}} k_i \right)^{1/2} \int \prod_{i \in \mathcal{N}} \frac{\mathrm{d}u_i}{\sqrt{2\pi/\mathrm{i}}} \exp\left\{ -\frac{\mathrm{i}}{2} \lambda_\varepsilon \sum_{i \in \mathcal{N}} k_i u_i^2 + \frac{\mathrm{i}}{2} \sum_{i,j \in \mathcal{N}} c_{ij} \ u_i u_j \right\} \ ,$$

Performing the Average

- To simplify, use canonical graph ensemle for given degree sequences

$$p(\{c_{ij}\}) = \prod_{i<j} \left[\left(1 - \frac{k_i k_j}{cN}\right) \delta_{c_{ij},0} + \frac{k_i k_j}{cN} \delta_{c_{ij},1} \right] \delta_{c_{ij},c_{ji}}$$

  for averaging over graph-ensemble.

- Get

$$\begin{aligned} \langle Z_{\mathcal{N}}^n \rangle \quad \propto \quad & \int \prod_{i\in\mathcal{N},a} \frac{\mathrm{d}u_{ia}}{\sqrt{2\pi/\mathrm{i}}} \exp\Bigg\{ -\frac{\mathrm{i}}{2} \lambda_\varepsilon \sum_{ia} k_i u_{ia}^2 \\ & + \frac{c}{2N} \sum_{i,j} \frac{k_i}{c} \frac{k_j}{c} \left(\exp\Big\{ i \sum_a u_{ia} u_{ja} \Big\} - 1 \right) \Bigg\} \end{aligned}$$

- To decouple sites, rewrite $\langle Z_{\mathcal{N}}^n \rangle$ as a functional integral, using the density

$$\rho(\mathbf{u}) = \frac{1}{N} \sum_{i\in\mathcal{N}} \frac{k_i}{c} \prod_a \delta(u_a - u_{ia})$$

Replica Symmetry

- Gives

$$\langle Z_{\mathcal{N}}^n \rangle \propto \int \mathcal{D}\{\rho,\hat{\rho}\} \exp\{N[G_b + G_m + G_s]\}$$

with

$$\begin{aligned}
G_b &= \frac{c}{2}\int \mathrm{d}\rho(\mathbf{u})\mathrm{d}\rho(\mathbf{v})\left(\exp\left\{i\sum_a u_a v_a\right\} - 1\right) \\
G_m &= -\int \mathrm{d}\mathbf{u}\ \mathrm{i}\hat{\rho}(\mathbf{u})\rho(\mathbf{u}) \\
G_s &= \sum_{k\geq 1} p(k) \ln \int \prod_a \frac{\mathrm{d}u_a}{\sqrt{2\pi/\mathrm{i}}} \exp\left\{-\frac{\mathrm{i}}{2}\lambda_\varepsilon k \sum_a u_a^2 + \mathrm{i}\frac{k}{c}\hat{\rho}(\mathbf{u})\right\}
\end{aligned}$$

- Replica symmetric ansatz (superpositions of complex Gaussians)

$$\rho(\mathbf{u}) = \int \mathrm{d}\pi(\omega) \prod_a \frac{\mathrm{e}^{-\frac{\omega}{2}u_a^2}}{Z(\omega)}\,, \qquad \mathrm{i}\hat{\rho}(\mathbf{u}) = \hat{c}\int \mathrm{d}\hat{\pi}(\hat{\omega}) \prod_a \frac{\mathrm{e}^{-\frac{\hat{\omega}}{2}u_a^2}}{Z(\hat{\omega})}$$

with $Z(\omega) = \sqrt{2\pi/\omega}$.

- Get self-consistency equations for weight functions $\pi(\omega)$ and $\hat{\pi}(\hat{\omega})$.

Self-Consistency Equations & Spectral Density

- Eliminate $\hat{\pi}(\hat{\omega})$ & revert to micro-canonical graph ensemble. Get

$$\pi(\omega) = \sum_{k\geq 1} p(k)\frac{k}{c}\int \prod_{\ell=1}^{k-1} \mathrm{d}\pi(\omega_\ell)\ \delta(\omega - \Omega_{k-1})$$

with

$$\Omega_{k-1} = \Omega_{k-1}(\{\omega_\ell\}) = \mathrm{i}\lambda_\varepsilon k + \sum_{\ell=1}^{k-1} \frac{1}{\omega_\ell}\ .$$

- Solve using stochastic (population dynamics) algorithm.
- In terms of these

$$\rho(\lambda) = p(0)\,\delta(\lambda - 1) + \frac{1}{\pi}\mathrm{Re}\sum_{k\geq 1} p(k)\int \prod_{\ell=1}^{k} \mathrm{d}\pi(\omega_\ell)\ \frac{k}{\Omega_k(\{\omega_\ell\})}$$

- Can identify continuous and pure point contributions to DOS.

Replica – Markov Matrices

- Replica analysis. Here for simplicity for symmetric rates from the start.
- Use decomposition $\bar{\mathcal{N}}, \mathcal{N}$ as before & rescale variables $u_i \leftarrow \frac{u_i}{\Gamma_i}$ on $\mathcal{N}$

$$Z_{\mathcal{N}} \propto \int \prod_{i\in\mathcal{N}} \frac{\mathrm{d}u_i}{\sqrt{2\pi/\mathrm{i}}} \exp\left\{ -\frac{\mathrm{i}}{2}\lambda_\varepsilon \sum_{i\in\mathcal{N}} \Gamma_i u_i^2 + \frac{\mathrm{i}}{2} \sum_{i,j\in\mathcal{N}} c_{ij}K_{ij}\, u_i u_j \right\}$$

- Rewrite

$$Z_{\mathcal{N}} \propto \int \prod_{i\in\mathcal{N}} \frac{\mathrm{d}u_i}{\sqrt{2\pi/\mathrm{i}}} \exp\left\{ -\frac{\mathrm{i}}{4}\lambda_\varepsilon \sum_{i,j\in\mathcal{N}} c_{ij}K_{ij}(u_i^2+u_j^2) + \frac{\mathrm{i}}{2} \sum_{i,j\in\mathcal{N}} c_{ij}K_{ij}\, u_i u_j \right\}$$

- Now average $Z_{\mathcal{N}}^n$ over c_{ij} and K_{ij}.
- Note different structure: eigenvalue now couples to bond disorder.

Performing the Average

- For simplicity use canonical graph-ensemle

$$
\begin{aligned}
\langle Z_{\mathcal{N}}^n \rangle \;\propto\; & \int \prod_{i\in\mathcal{N},a} \frac{\mathrm{d}u_{ia}}{\sqrt{2\pi/\mathrm{i}}} \exp\Bigg\{ \frac{c}{2N} \sum_{i,j} \frac{k_i}{c}\frac{k_j}{c} \\
& \times \left(\left\langle \exp\left\{ iK \sum_a \left[u_{ia}u_{ja} - \frac{\mathrm{i}}{2}\lambda_\varepsilon (u_{ia}^2 + u_{ja}^2) \right] \right\} \right\rangle_K - 1 \right) \Bigg\}
\end{aligned}
$$

- Rewrite this as a functional integral over replica-densities as before.

$$
\langle Z_{\mathcal{N}}^n \rangle \propto \int \mathcal{D}\{\rho,\hat\rho\} \exp\{N[G_b + G_m + G_s]\}
$$

with

$$
\begin{aligned}
G_b &= \frac{c}{2} \int \mathrm{d}\rho(\mathbf{u})\mathrm{d}\rho(\mathbf{v}) \left(\left\langle \exp\left\{ iK \sum_a \left[u_a v_a - \frac{1}{2}\lambda_\varepsilon (u_a^2 + v_a^2) \right] \right\} \right\rangle_K - 1 \right) \\
G_m &= -\int \mathrm{d}\mathbf{u}\; \mathrm{i}\hat\rho(\mathbf{u})\rho(\mathbf{u}) \\
G_s &= \sum_{k\geq 1} p(k) \ln \int \prod_a \frac{\mathrm{d}u_a}{\sqrt{2\pi/\mathrm{i}}} \exp\left\{ \mathrm{i}\frac{k}{c}\hat\rho(\mathbf{u}) \right\}
\end{aligned}
$$

Replica Symmetry

- Replica symmetric ansatz (superpositions of complex Gaussians)

$$\rho(\mathbf{u}) = \int d\pi(\omega) \prod_a \frac{e^{-\frac{\omega}{2}u_a^2}}{Z(\omega)}\,, \qquad i\hat{\rho}(\mathbf{u}) = \hat{c}\int d\hat{\pi}(\hat{\omega}) \prod_a \frac{e^{-\frac{\hat{\omega}}{2}u_a^2}}{Z(\hat{\omega})}$$

- (micro-canonical limit)

$$G_b \simeq n\frac{c}{2}\int d\pi(\omega)d\pi(\omega') \left\langle \ln\left[\frac{Z_2(\omega,\omega',K,\lambda_\varepsilon)}{Z(\omega)Z(\omega')}\right]\right\rangle_K$$

$$G_m \simeq -\hat{c} - n\hat{c}\int d\pi(\omega)d\hat{\pi}(\hat{\omega}) \ln\left[\frac{Z(\omega+\hat{\omega})}{Z(\omega)Z(\hat{\omega})}\right]$$

$$G_s \simeq \hat{c} + n\sum_{k\geq 1} p(k)\int \prod_{\nu=1}^{k} d\hat{\pi}(\hat{\omega}_\nu) \ln\left[\frac{Z(\sum_{\nu=1}^{k}\hat{\omega}_\nu)}{\sqrt{2\pi/i}\prod_{\nu=1}^{k} Z(\hat{\omega}_\nu)}\right]$$

with $Z(\omega) = \sqrt{2\pi/\omega}$, and

$$Z_2(\omega,\omega',K,\lambda_\varepsilon) = Z(\omega' + i\lambda_\varepsilon K)\, Z\left(\omega + i\lambda_\varepsilon K + \frac{K^2}{\omega' + i\lambda_\varepsilon K}\right).$$

Replica Symmetry – Self-Consistency Equations

- FPEs from stationarity conditions (after eliminating $\hat{\pi}(\hat{\omega})$)

$$\pi(\omega) = \sum_{k\geq 1} p(k)\frac{k}{c}\int \prod_{\nu=1}^{k-1} d\pi(\omega_\nu) \ \left\langle \delta\left(\omega - \Omega_{k-1}\right)\right\rangle_{\{K_\nu\}}$$

with

$$\Omega_{k-1} = \sum_{\nu=1}^{k-1}\left[i\lambda_\varepsilon K_\nu + \frac{K_\nu^2}{\omega_\nu + i\lambda_\varepsilon K_\nu}\right] .$$

- Spectral density now involves derivative of bond-related part G_b w.r.t $\lambda \Rightarrow$ Immediate interpretation in terms of sum over site (Trace) is lost. (Can recover using FPEs.)

$$\rho(\lambda) = \frac{c}{2\pi(1-p(0))}\mathrm{Re}\int d\pi(\omega)d\pi(\omega')\left\langle \frac{K(\omega+\omega') + 2i\lambda_\varepsilon K^2}{(i\lambda_\varepsilon K + \omega)(i\lambda_\varepsilon K + \omega') + K^2}\right\rangle_K$$

Analytically Tractable Limiting Cases

Unbiased Random Walk on Random Regular Graph

- Recall FPE

$$\pi(\omega) = \sum_{k\geq 1} p(k)\frac{k}{c}\int \prod_{\ell=1}^{k-1} \mathrm{d}\pi(\omega_\ell)\ \delta(\omega - \Omega_{k-1})$$

with

$$\Omega_{k-1} = \mathrm{i}\lambda_\varepsilon k + \sum_{\ell=1}^{k-1}\frac{1}{\omega_\ell}\ .$$

- Regular Random Graphs $p(K) = \delta_{k,c}$. All sites equivalent.
- $\Rightarrow$ Expect

$$\pi(\omega) = \delta(\omega - \bar{\omega})\ , \qquad \Leftrightarrow \qquad \bar{\omega} = \mathrm{i}\lambda_\varepsilon c + \frac{c-1}{\bar{\omega}}$$

- Gives

$$\rho(\lambda) = \frac{c}{2\pi}\frac{\sqrt{4\frac{c-1}{c^2} - \lambda^2}}{1-\lambda^2}$$

- $\Leftrightarrow$ Kesten-McKay distribution adapted to Markov matrices

Analytically Tractable Limiting Cases

Unbiased Random Walk on Large-c Erdös-Renyi Graph

- For large-c Erdös-Renyi Graph, $p(k)$ gives weight to k-range $k = c \pm O(\sqrt{c})$. Relative fluctuations of degree become negligible.
- Asymptotically at large c all sites effectively equivalent.
- $\Rightarrow$ Expect

$$\pi(\omega) \simeq \delta(\omega - \bar{\omega}) \ , \qquad \Leftrightarrow \qquad \bar{\omega} = \mathrm{i}\lambda_\varepsilon c + \frac{c-1}{\bar{\omega}}$$

  to become exact for $c \gg 1$.
- Gives

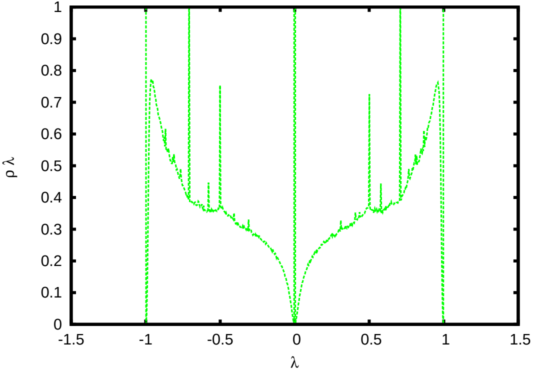
$$\rho(\lambda) = \frac{c}{2\pi} \frac{\sqrt{4\frac{c-1}{c^2} - \lambda^2}}{1-\lambda^2}$$

  as before but now to be used in large-c-limit! $\Rightarrow$ Wigner semi-circle!

Outline

Unbiased Random Walk

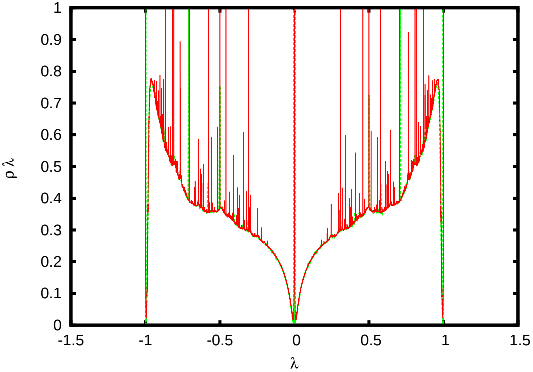
- Spectral density: $k_i \sim \text{Poisson}(2)$, $\mathcal{W}$ unbiased RW



Simulation results, averaged over 5000 1000×1000 matrices (green);

Unbiased Random Walk

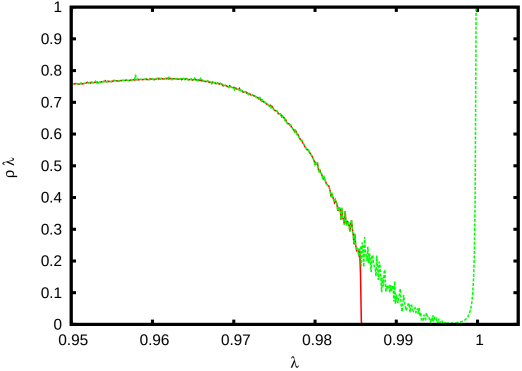
- Spectral density: $k_i \sim$ Poisson(2), $\mathcal{W}$ unbiased RW



Simulation results, averaged over 5000 1000×1000 matrices (green); population-dynamics results (red) added;

Unbiased Random Walk

- Spectral density: $k_i \sim$ Poisson(2), $\mathcal{W}$ unbiased RW

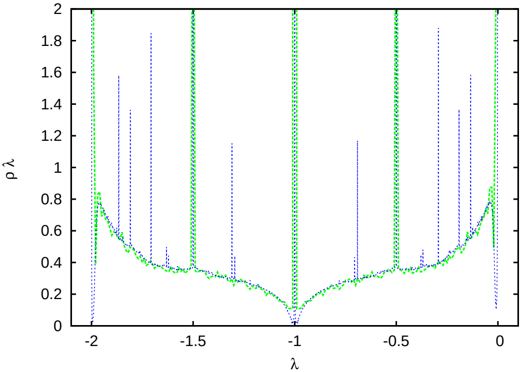


Simulation results, averaged over 5000 1000×1000 matrices (green); population-dynamics results (red) added;

population dynamics results: zoom into $\lambda \simeq 1$ region. (total DOS green, extended states (red).

Unbiased Random Walk

- comparison population dynamics – cavity on single instance $k_i \sim$ Poisson(2)

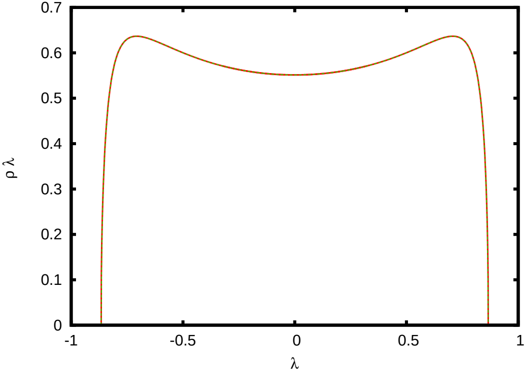


Population dynamics results (blue) compared to results from cavity approach

on a single instance of $N = 10^4$ sites (green), both for total DOS – Normalized Graph Laplacian

Unbiased Random Walk–Regular Random Graph

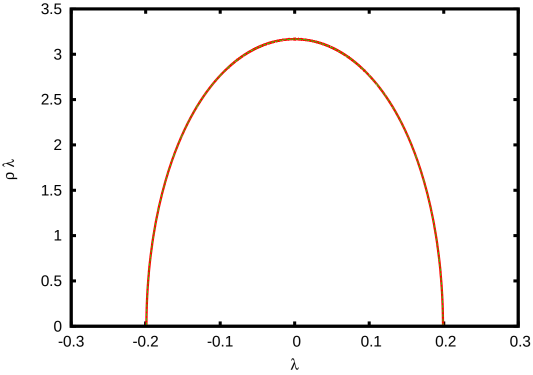
- comparison population dynamics – analytic result



Population dynamics results (red) compared to analytic result (green) for RW on regular random graph at $c = 4$.

Unbiased Random Walk–Large=c Erdös-Renyi

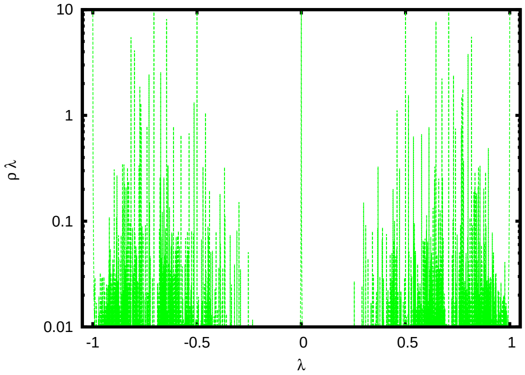
- comparison population dynamics – analytic result



Population dynamics results (red) compared to analytic result (green) for RW on Erdös-Renyi random graph at $c = 100$.

Unbiased Random Walk–Scale Free Graphs

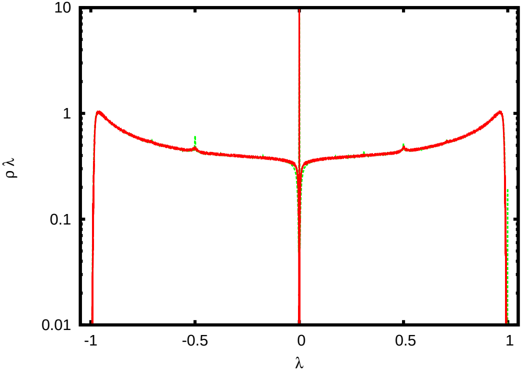
- Random graphs with $p(k) \propto k^{-\gamma}$, $k \geq k_{\min}$



Population dynamics results for RW on scale-free graph $\gamma = 4$, $k_{\min} = 1$.

Unbiased Random Walk–Scale Free Graphs

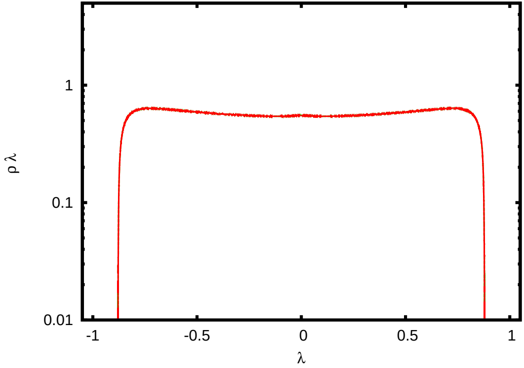
- Random graphs with $p(k) \propto k^{-\gamma}, k \geq k_{\min}$



Simulation results (green) compared with population dynamics results (red) for a RW on scale-free graph $\gamma = 4, k_{\min} = 2$.

Unbiased Random Walk–Scale Free Graphs

- Random graphs with $p(k) \propto k^{-\gamma}$, $k \geq k_{\min}$

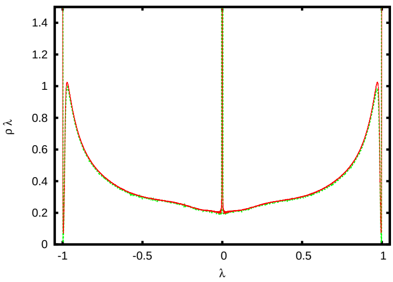


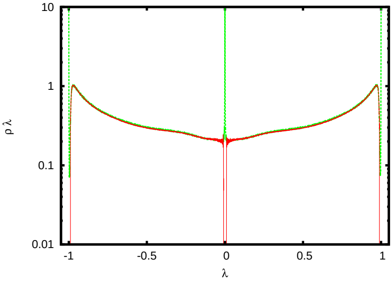
Population dynamics results (extende DOS red, total DOS green) for a RW on scale-free graph $\gamma = 4$, $k_{\min} = 3$.

Stochastic Matrices

- Spectral density: $k_i \sim \text{Poisson}(2)$, $p(K_{ij}) \propto K_{ij}^{-1}$; $K_{ij} \in [\mathrm{e}^{-\beta}, 1]$

$$\Leftrightarrow K_{ij} = exp\{-\beta V_{ij}\} \text{ with } V_{ij} \sim U[0,1] \Leftrightarrow \text{ Kramers rates.}$$



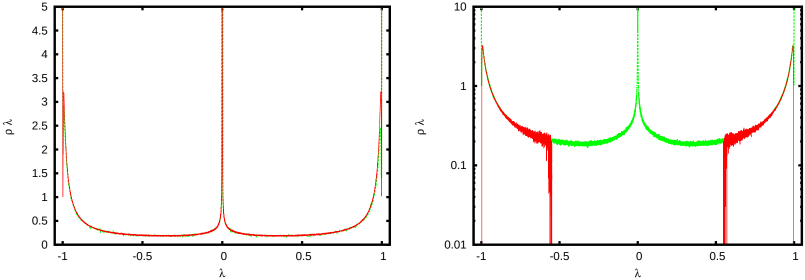


Spectral density for stochastic matrices defined on Poisson random graphs with $c = 2$, and $\beta = 2$. Left: Simulation results (green) compared with population dynamics results; Right Population dynamics results (total DOS red); extended states (green).

Stochastic Matrices

- Spectral density: $k_i \sim \text{Poisson}(2)$, $p(K_{ij}) \propto K_{ij}^{-1}$; $K_{ij} \in [\mathrm{e}^{-\beta}, 1]$

$$\Leftrightarrow K_{ij} = exp\{-\beta V_{ij}\} \text{ with } V_{ij} \sim U[0,1] \Leftrightarrow \text{ Kramers rates.}$$

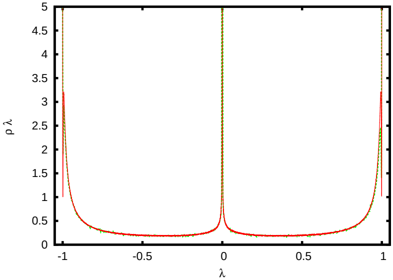


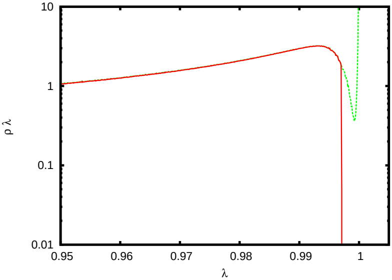
Spectral density for stochastic matrices defined on Poisson random graphs with $c = 2$, and $\beta = 5$. Left: Simulation results (green) compared with population dynamics results (red); Right Population dynamics results (total DOS green); extended states (red).

Stochastic Matrices

- Spectral density: $k_i \sim \text{Poisson}(2)$, $p(K_{ij}) \propto K_{ij}^{-1}$; $K_{ij} \in [\mathrm{e}^{-\beta}, 1]$

  $\Leftrightarrow K_{ij} = exp\{-\beta V_{ij}\}$ with $V_{ij} \sim U[0,1] \Leftrightarrow$ Kramers rates.



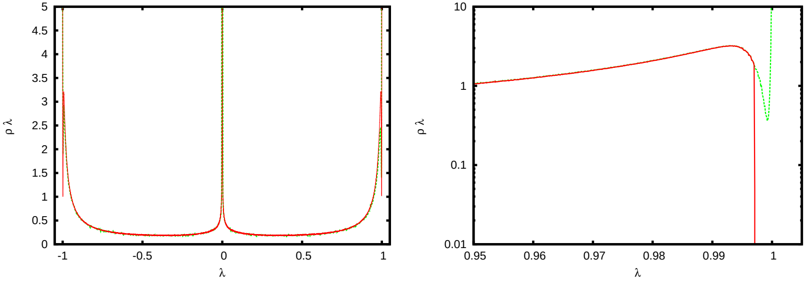


Spectral density for stochastic matrices defined on Poisson random graphs with $c = 2$, and $\beta = 5$. Left: Simulation results (green) compared with population dynamics results (red); Right Population dynamics results (total DOS green); extended states (red) for $\lambda \simeq 1$.

Stochastic Matrices

- Spectral density: $k_i \sim$ Poisson(2), $p(K_{ij}) \propto K_{ij}^{-1}; K_{ij} \in [\mathrm{e}^{-\beta}, 1]$

$$\Leftrightarrow K_{ij} = exp\{-\beta V_{ij}\} \text{ with } V_{ij} \sim U[0,1] \Leftrightarrow \text{ Kramers rates.}$$



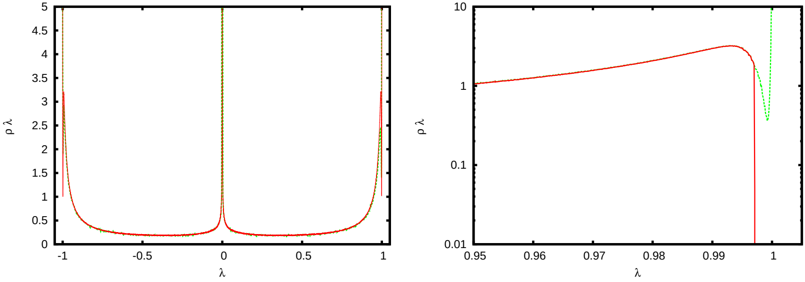
Spectral density for stochastic matrices defined on Poisson random graphs with $c = 2$, and $\beta = 5$. Left: Simulation results (green) compared with population dynamics results (red); Right Population dynamics results (total DOS green); extended states (red) for $\lambda \simeq 1$.

- Density of states concentrates at edges of spectrum as β increases (slow modes!)

Stochastic Matrices

- Spectral density: $k_i \sim$ Poisson(2), $p(K_{ij}) \propto K_{ij}^{-1}$; $K_{ij} \in [e^{-\beta}, 1]$

$$\Leftrightarrow K_{ij} = exp\{-\beta V_{ij}\} \text{ with } V_{ij} \sim U[0,1] \Leftrightarrow \text{ Kramers rates.}$$

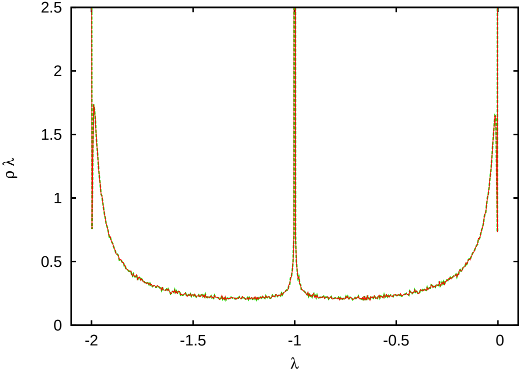


Spectral density for stochastic matrices defined on Poisson random graphs with $c = 2$, and $\beta = 5$. Left: Simulation results (green) compared with population dynamics results (red); Right Population dynamics results (total DOS green); extended states (red) for $\lambda \simeq 1$.

- Density of states concentrates at edges of spectrum as β increases (slow modes!)
- Contribution of localized state to total DOS in bulk of specturm increases as β increases.

Stochastic Matrices

- Spectral density: $k_i \sim \text{Poisson}(2)$, $p(K_{ij}) = T^{-1}\exp(-K_{ij}/T)$.



Spectral density for stochastic matrices defined on Poisson random graphs with $c = 2$.

Unnormalized weights $K_{ij} = -T\log(r_{ij})$ with $r_{ij} \sim \mathrm{U}[0,1]$. Simulation results for $T = 2$ (green) and $T = 5$ (red).

- Spectra independent of T! (Similarly: for $K_{ij} \sim U[0, K_{\max}]$, spectra independent of $K_{\max}$)

Outline

Summary

- Computed DOS of Stochastic matrices defined on random graphs
- Localized states at edges of specrum implies finite maximal relaxation time even in thermodynamic limit
  (related to minimal metallic conductivity ?)
- For $p(K_{ij}) = T^{-1}\exp(-K_{ij}/T)$, spectra independent of mean T.
- Similarly, for $p(K_{ij}) = \mathrm{const.}$ on $[0, K_{\mathrm{max}}]$, no dependence on K_{max}.
- For $p(K_{ij}) \propto K_{ij}^{-1}$; $K_{ij} \in [\mathrm{e}^{-\beta}, 1]$ see localization effects at large β and concetration of DOS at edges of the spectrum ($\leftrightarrow$ relaxation time spectrum dominated by slow modes $\Rightarrow$ Glassy Dynamics?

Cavity

- Alternatively use the cavity method to evaluate the marginals

$$P(u_i) \propto \exp\Big\{-\frac{\mathrm{i}}{2}\lambda_\varepsilon k_i u_i^2\Big\} \int \prod_{j\in\partial i} \mathrm{d}u_j \exp\Big\{\mathrm{i}\sum_{j\in\partial i} u_i u_j\Big\} P_j^{(i)}(u_j) ,$$

- On a (locally) tree-like graph one may write down a recursion for the cavity distributions,

$$P_j^{(i)}(u_j) \propto \exp\Big\{-\frac{\mathrm{i}}{2}\lambda_\varepsilon k_j u_j^2\Big\} \prod_{\ell\in\partial j\setminus i} \int \mathrm{d}u_\ell \exp\Big\{\mathrm{i} u_j u_\ell\Big\} P_\ell^{(j)}(u_\ell) .$$

- Recursions of this type are self-consistently solved by complex Gaussians of the form

$$P_j^{(i)}(u_j) = \sqrt{\frac{\omega_j^{(i)}}{2\pi}} \exp\Big\{-\frac{1}{2}\omega_j^{(i)} u_j^2\Big\} ,$$

Cavity Recursion and Spectral Density

- Generates a recursion for the $\omega_j^{(i)}$ of the form

$$\omega_j^{(i)} = \mathrm{i}\lambda_\varepsilon k_j + \sum_{\ell \in \partial j \setminus i} \frac{1}{\omega_\ell^{(j)}}$$

- Solve iteratively on single large instances; use to evaluate

$$\rho_N(\lambda) = p(0)\delta(\lambda - 1) + \frac{1}{\pi N} \mathrm{Re} \sum_{i \in \mathcal{N}} k_i \langle u_i^2 \rangle \ ,$$

with

$$\langle u_i^2 \rangle = \frac{1}{\mathrm{i}\lambda_\varepsilon k_i + \sum_{j \in \partial i} \frac{1}{\omega_j^{(i)}}}$$

Cavity – Markov Matrices

- Repeat analysis for master-equation operators other than normalized graph Laplacian.
- Cavity equations for single instances obtained with obvious minor modifications: for $i \in \mathcal{N}$

$$P_j^{(i)}(u_j) \propto \exp\left\{-\frac{\mathrm{i}}{2}\lambda_\varepsilon \Gamma_j u_j^2\right\} \prod_{\ell \in \partial j \setminus i} \int \mathrm{d}u_\ell \exp\left\{\mathrm{i}\Gamma_{j\ell} u_j u_\ell\right\} P_\ell^{(j)}(u_\ell) \ .$$

- Generates a recursion for the $\omega_j^{(i)}$ of the form

$$\omega_j^{(i)} = \mathrm{i}\lambda_\varepsilon \Gamma_j + \sum_{\ell \in \partial j \setminus i} \frac{\Gamma_{j\ell}^2}{\omega_\ell^{(j)}}$$

- Solve iteratively on single large instances; use to evaluate

$$\rho_N(\lambda) = p(0)\delta(\lambda - 1) + \frac{1}{\pi N} \mathrm{Re} \sum_{i \in \mathcal{N}} \Gamma_i \langle u_i^2 \rangle \ ,$$