Arithmetic theta series

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Workshop on Representation Theory and Analysis on Locally Symmetric Spaces
IAS
March 8, 2018

Introduction

The theta correspondence between automorphic representations of groups (H, G) in a reductive dual pair, as well as its local version, has proved to be a useful tool in the study of such representations.

The basis for this correspondence is the use of theta functions $\theta(h, g; \varphi)$ built from Schwartz functions φ, say in some Schrödinger model of the Weil representation, as integral kernels to transport cuspidal automorphic functions from one group to the other.

The seesaw identities, Siegel-Weil formula, and the doubling method then yield criteria for the non-vanishing of such theta lifts in terms of special values of L-functions and local obstructions.

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As a variant of this, when the group G is a classical group $O(p, q)$, $U(p, q)$ or $\mathrm{Sp}(p, q)$, Millson and I constructed theta functions valued in the deRham complex for the associated locally symmetric manifold, $M = \Gamma \backslash D$.

These 'geometric' theta series are closely linked to a certain type of locally symmetric cycles in M.

The geometric theta series are closed as differential forms and, passing to cohomology, they give rise to a theta correspondence between automorphic forms on H and cohomology classes on M.

Such correspondences are the starting point for many applications, among them the recent striking results of Bergeron-Millson-Moeglin (2014).

Among other things, they establish new cases of the Hodge conjecture for certain ball quotients where M is a quasi-projective variety and the locally symmetric cycles are, in fact, algebraic cycles on M.

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Very roughly, these arithmetic theta series should arise in the case where M is a Shimura variety with a regular integral model $\mathcal{M}$.

The idea is to constructed generating series for classes of certain 'special cycles' in $\mathcal{M}$ in the arithmetic Chow groups $\widehat{\mathrm{CH}}^r(\mathcal{M})$.

The goal is then to show that these series define $\widehat{\mathrm{CH}}^r(\mathcal{M})$-valued automorphic forms on H.

These arithmetic theta series would then provide an arithmetic theta correspondence between automorphic forms on H and classes in the arithmetic Chow groups.

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The unitary Shimura variety

Here is some notation:

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\begin{aligned}
\Bbbk &= \text{imaginary quadratic field with odd discr. } -D \\
W &= \text{hermitian space over } \Bbbk \text{ of signature } (n-1,1) \\
W_0 &= \text{hermitian space over } \Bbbk \text{ of signature } (1,0) \\
G &= \{(g_0, g) \in \mathrm{GU}(W_0) \times \mathrm{GU}(W) \mid \nu(g_0) = \nu(g)\ \} \\
\mathfrak{a}, \mathfrak{a}_0 &= \text{self-dual } O_{\Bbbk}\text{-lattices in } W \text{ and } W_0 \\
K &= G(\mathbb{A}_f) \cap (\, K_{\mathfrak{a}_0} \times K_{\mathfrak{a}}\,), \quad \text{compact open} \\
\mathrm{Sh}(G, \mathcal{D})(\mathbb{C}) &= G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K, \quad \text{the Shimura variety.}
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The presence of the 'extra' factor coming from W_0 is essential in the definition of the special cycles.

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The presence of the ‘extra’ factor coming from W_0 is essential in the definition of the special cycles.

A modular interpretation

To define the integral model, consider the following moduli problem:

To an O_k-scheme S assign the groupoid of triples (A, ι, ψ) where

$$A \longrightarrow S \quad \text{an abelian scheme of relative dim. } n$$

$$\iota : O_k \longrightarrow \mathrm{End}(A) \quad \text{an } O_k \text{ action such that}$$

$$\det(T - \iota(\alpha)|\mathrm{Lie}\,(A)) = (T - \alpha)^{n-1}(T - \bar{\alpha}) \in \mathcal{O}_S[T],$$

$$\psi : A \longrightarrow A^\vee \quad \text{a principal polarization such that}$$

$$\iota(\alpha)^\dagger = \iota(\bar{\alpha}). \qquad \dagger = \text{Rosati for } \psi.$$

This a not quite good enough at primes dividing the discriminant of k.

To obtain a better model, enhance the data to $(A, \iota, \psi, \mathcal{F}_A)$, where

$$\mathcal{F}_A \subset \mathrm{Lie}\,(A) = O_k\text{-stable, locally direct } \mathcal{O}_S \text{ submodule of rank } n-1$$

with O_k-acting on $\mathcal{F}_A$ via $O_k \longrightarrow \mathcal{O}_S$, and
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The resulting moduli stack[3] $\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)}$ over $\operatorname{Spec} O_{\Bbbk}$ is regular and flat.

The moduli stack $\mathcal{M}_{(1,0)}$ over $\operatorname{Spec} O_{\Bbbk}$ defined via triples (A_0, ι_0, ψ_0) as above is already smooth[4] over $\operatorname{Spec} O_{\Bbbk}$.

If we denote the generic fibers of these stacks by $M_{(n-1,1)}$ and $M_{(1,0)}$, then

$$\mathrm{Sh}(G, \mathcal{D}) \subset M_{(1,0)} \times_{\Bbbk} M_{(n-1,1)}$$

is an open and closed substack,

characterized by the existence of an isomorphism, for every prime ℓ,

$$\mathrm{Hom}_{O_{\Bbbk}}(T_\ell A_{0,s}, T_\ell A_s) \simeq \mathrm{Hom}_{O_{\Bbbk}}(\mathfrak{a}_0, \mathfrak{a}) \otimes \mathbb{Z}_\ell$$

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$$\mathrm{Hom}_{O_k}(T_\ell A_{0,s}, T_\ell A_s) \simeq \mathrm{Hom}_{O_k}(\mathfrak{a}_0, \mathfrak{a}) \otimes \mathbb{Z}_\ell$$

at every geometric point s. This is required to be an isometry for the natural $(O_k)_\ell$-hermitian form on the two sides.

[3] the Krämer model

[4] Howard (2015)

A modular interpretation

The resulting moduli stack[3] $\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)}$ over $\mathrm{Spec}\, O_{\Bbbk}$ is regular and flat.

The moduli stack $\mathcal{M}_{(1,0)}$ over $\mathrm{Spec}\, O_{\Bbbk}$ defined via triples (A_0, ι_0, ψ_0) as above is already smooth[4] over $\mathrm{Spec}\, O_{\Bbbk}$.

If we denote the generic fibers of these stacks by $M_{(n-1,1)}$ and $M_{(1,0)}$, then

$$\mathrm{Sh}(G, \mathcal{D}) \ \subset \ M_{(1,0)} \times_{\Bbbk} M_{(n-1,1)}$$

is an open and closed substack,

characterized by the existence of an isomorphism, for every prime ℓ,

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A modular interpretation

We obtain our integral model by taking[5] the Zariski closure

$$\begin{tikzcd}
\mathrm{Exc} \arrow[r] \arrow[d] & \mathcal{S}_{\mathrm{Kra}} \arrow[r] \arrow[d] & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \arrow[d] & \\
\mathrm{Sing} \arrow[r] & \mathcal{S}_{\mathrm{Pap}} \arrow[r] & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} & \nwarrow \text{ blowup} \\
 & \mathrm{Sh}(G,\mathcal{D}) \arrow[r] \arrow[u] & M_{(1,0)} \times M_{(n-1,1)} \arrow[u] & \longleftarrow \text{ Zariski closure}
\end{tikzcd}$$

Here $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ is an intermediate model defined by adding the Pappas wedge condition rather than the Krämer condition. It has isolated singular points in fibers over ramified primes. These are blown up to an exceptional divisor in the Krämer model.

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Special divisors in the integral models

The upshot of the previous discussion is that we have nice integral models

$$\mathcal{S}_{\mathsf{Kra}} \longrightarrow \mathcal{S}_{\mathsf{Pap}},$$

related by a blowup.

They both have nice toroidal compactifications

$$\mathcal{S}^*_{\mathsf{Kra}} \longrightarrow \mathcal{S}^*_{\mathsf{Pap}},$$

with boundary divisors to be discussed in a moment.

In a careful treatment, the two must be carried along since:

- $\mathcal{S}^*_{\mathsf{Pap}}$ is not regular but every vertical Weil divisor meets the boundary.
- $\mathcal{S}^*_{\mathsf{Kra}}$ is regular but Exc does not meet the boundary.

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Special divisors in the integral models

For an $O_{\Bbbk}$-scheme S, an S-valued point of $\mathcal{S}$ corresponds to a pair (A_0, A) of principally polarized abelian schemes with $O_{\Bbbk}$-action and some additional equipment.

For such a pair, the $O_{\Bbbk}$-lattice

$$L(A_0, A) = \mathrm{Hom}_{O_{\Bbbk}}(A_0, A)$$

has a natural positive definite hermitian form defined by

$$\langle x_1, x_2 \rangle = x_2^\vee \circ x_1 \in \mathrm{End}_{O_{\Bbbk}}(A_0) \xrightarrow[\sim]{\iota_0^{-1}} O_{\Bbbk},$$

$$\begin{tikzcd} A_0 \arrow[r, "x_1"] & A \arrow[d, "\psi"] \\ A_0^\vee \arrow[u, "\psi_0^{-1}"] & A^\vee . \arrow[l, "x_2^\vee"] \end{tikzcd}$$

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For $m \in \mathbb{Z}_{>0}$, let

$$\mathcal{Z}(m)(S) = \begin{pmatrix} \text{groupoid of triples } (A_0, A, x), \\ (A_0, A) \in \mathcal{S}(S) \\ x \in L(A_0, A), \text{ with } \langle x, x \rangle = m \end{pmatrix} \longrightarrow \mathcal{S}(S).$$

The arithmetic special divisors $\mathcal{Z}(m)$'s are Cartier divisors on $\mathcal{S}$.

Conceptually, they are the loci where the abelian variety A is equipped with an elliptic curve factor A_0.

Let $\mathcal{Z}^*(m)$ be the Zariski closure of $\mathcal{Z}(m)$ in $\mathcal{S}^*$, the toroidal compactification of $\mathcal{S}$.

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Structure of the compactification

It is now time to say something about the compactification $\mathcal{S}^*$.

Recall that our Shimura variety is

$$\mathrm{Sh}(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K,$$

where

$$\mathcal{D} = \text{negative lines in } V_{\mathbb{R}},$$

and

$$V = \mathrm{Hom}_{\Bbbk}(W_0, W), \qquad \langle x, y \rangle = y^{\vee} \circ x.$$

Note that the definition of V, where the extra hermitian space W_0 seems unnecessary, is motivated by the definition of the special cycles where the role of the 'auxiliary' elliptic curve A_0 is essential.

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The cusps of $\mathrm{Sh}(G, \mathcal{D})(\mathbb{C})$ are then indexed by pairs $\Phi = (J, g)$, with $J \subset V$ an isotropic k-line and $g \in G(\mathbb{A}_f)$, modulo a suitable equivalence.

Associated to Φ is a filtration

$$0 \subset J \subset J^{\perp} \subset V$$

and an integral version

$$0 \ \subset \ J \cap gL \ \subset \ J^{\perp} \cap gL \ \subset \ gL = \mathrm{Hom}_{\mathcal{O}_k}(g\mathfrak{a}_0, g\mathfrak{a}).$$

The hermitian lattice

$$L_{\Phi} := (gL \cap J^{\perp})/gL \cap J$$

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The toroidal compactification of $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$ is then obtained by blowing up the cusps in the minimal compactification

$$\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})^{\mathrm{BB}} \hookrightarrow M_{(1,0)}(\mathbb{C}) \times M^{\mathrm{BB}}_{(n-1,0)}$$

to the abelian varieties $\mathcal{B}_\Phi(\mathbb{C})$ of dimension $n-2$, where

$$\mathcal{B}_\Phi = E \otimes_{O_k} L_\Phi,$$

for $E \longrightarrow \mathcal{M}_{(1,0)}$ the universal CM-elliptic scheme.
This picture propagates to the integral model.
For the finite isometry group

$$\Delta_\Phi = O_k^\times \times \mathrm{U}(L_\Phi),$$

define the orbifold quotient

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$$\mathcal{S}^* = \mathcal{S} \sqcup \bigsqcup_{\substack{\Phi \\ \text{mod} \sim}} \mathcal{S}^*(\Phi),$$

where the boundary is a smooth divisor, flat over O_k.

We next augment the divisor $\mathcal{Z}^*(m)$ by adding a rational linear combination of boundary divisors:

$$\mathcal{Z}^{\text{tot}}(m) := \mathcal{Z}^*(m) + \mathcal{BD}(m),$$

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The generating series

We are almost ready to complete the definition of the arithmetic theta series

$$\widehat{\phi}(\tau) = \widehat{\mathcal{Z}}^{\mathrm{tot}}(0) + \sum_{m=1}^{\infty} \widehat{\mathcal{Z}}^{\mathrm{tot}}(m)\, \mathbf{q}^m \;\in\; \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*).$$

The remaining issues are:

(1) The addition of the Green functions needed to define classes in the arithmetic Chow group $\widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*)$, and

(2) the definition of the constant term $\widehat{\mathcal{Z}}^{\mathrm{tot}}(0)$.

Here recall that classes in $\widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*)$ are given by pairs $(\mathcal{Z}, g_{\mathcal{Z}})$ where $\mathcal{Z}$ is a divisor on $\mathcal{S}^*$ and $g_{\mathcal{Z}}$ is a Green function on $\mathcal{S}^*(\mathbb{C}) \setminus \mathcal{Z}(\mathbb{C})$ with

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Relations are given by $(\mathrm{div}\psi, -\log|\psi|^2)$, for rational functions ψ on $\mathcal{S}^*$.

The generating series

We are almost ready to complete the definition of the arithmetic theta series

$$\widehat{\phi}(\tau) = \widehat{\mathcal{Z}}^{\mathrm{tot}}(0) + \sum_{m=1}^{\infty} \widehat{\mathcal{Z}}^{\mathrm{tot}}(m)\, \mathbf{q}^m \ \in\ \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*).$$

The remaining issues are:

(1) The addition of the Green functions needed to define classes in the arithmetic Chow group $\widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*)$, and

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To define $\mathcal{Z}(0)$, note that

$$\mathcal{D} \hookrightarrow \mathbb{P}(V_{\mathbb{R}}),$$

and denote by ω_{an} the pullback to $\mathcal{D}$ of the tautological bundle.

Note that this bundle comes with a hermitian metric defined by the restriction of

$$-\frac{1}{4\pi}e^{-\gamma}\langle\ ,\ \rangle_V,$$

where $\gamma = -\Gamma'(1)$ is Euler's constant[6].

A natural extension of (the inverse of) ω_{an} to $\mathcal{S} = \mathcal{S}_{\mathrm{Kra}}$ is defined by

$$\omega^{-1} = \mathrm{Lie}\,(A_0) \otimes \mathrm{Lie}\,(A)/\mathcal{F}_A.$$

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Finally, the metric on ω extends to a metric with log $-$ log singularities along the boundary[7] of $\mathcal{S}^*(\mathbb{C}) \setminus \mathcal{S}(\mathbb{C})$, so we obtain a class[8]

$$\widehat{\omega} = (\omega, ||\cdot||) \in \widehat{\mathrm{Pic}}_{\mathbb{Q}}(\mathcal{S}^*) \simeq \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*).$$

Define

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}(0) := \widehat{\omega}^{-1} + (\mathrm{Exc}, -\log D) \in \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*).$$

Also, for $m > 0$, define

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}(m) = (\mathcal{Z}^{\mathrm{tot}}(m), \Theta^{\mathrm{reg}}(f_m)) \in \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*),$$

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The main theorem

Aside from the detailed definition of the Green functions, we have in hand the complete definition of the arithmetic theta series

$$\widehat{\phi}(\tau) = \widehat{\mathcal{Z}}^{\mathrm{tot}}(0) + \sum_{m=1}^{\infty} \widehat{\mathcal{Z}}^{\mathrm{tot}}(m)\,\mathbf{q}^m \;\in\; \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*)[[\mathbf{q}]].$$

Main Theorem (BHKRY)**.** The formal series $\widehat{\phi}(\tau)$ is a $\widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*)$-valued modular form of weight n, level D, and character $\chi = \chi_{\boldsymbol{k}}^n$.

This means that for any $\mathbb{Q}$-linear function $\alpha : \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*) \longrightarrow \mathbb{C}$, the series

$$\sum_{m \geq 0} \alpha\big(\widehat{\mathcal{Z}}^{\mathrm{tot}}(m)\big)\,\mathbf{q}^m$$

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Consequences

As a formal consequence we have:

Corollary. The dimension of the subspace of $\widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*)$ spanned by the classes $\widehat{\mathcal{Z}}^{\mathrm{tot}}(m)$, for $m \geq 0$ is at most $\dim M_n(D, \chi_k^n)$.

Another important consequence is that we can define an *arithmetic theta lift*

$$\widehat{\theta} : S_n(\Gamma_0(D), \chi_k^n) \longrightarrow \widehat{\mathrm{CH}}^1(\mathcal{S}^*), \qquad f \mapsto \widehat{\theta}(f) = \langle\, \widehat{\phi}, f\, \rangle_{\mathrm{Pet}}.$$

Various constructions analogous to the more familiar ones involving the classical theta correspondence, seesaw identities, etc., give rise to expressions relating height pairings to special values.

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Modularity criterion

The basic idea is to use the 'duality method' introduced by Borcherds.

Modularity criterion. For $k \geq 2$, and for a formal power series

$$\phi(\mathbf{q}) = \sum_{m \geq 0} d(m)\, \mathbf{q}^m \;\in\; \mathbb{C}[[\mathbf{q}]],$$

the following are equivalent:

(1) $\phi(\mathbf{q})$ is the q-expansion of a modular form[9] in $M_k^{\infty}(D,\chi)$.

(2) the relation

$$\sum_{m \geq 0} c(-m)\, d(m) = 0$$

for every weakly holomorphic form[10]

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Those familiar with the theory of Borcherds forms will recognize that, as a starting point, we will want to associate to such a weakly homomorphic form f, with $c(-m)\in\mathbb{Z}$ for $m\geq 0$, a meromorphic section $\psi(f)$ of ω^k_{an} on $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$.

This is the unitary group analogue of the Borcherds lift for $\mathrm{SO}(n-2,2)$.

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Unitary Borcherds forms

Indeed, we have a morphism of Shimura varieties

$$j : \mathrm{Sh}(G, \mathcal{D})(\mathbb{C}) \longrightarrow \mathrm{Sh}(\widetilde{G}, \widetilde{\mathcal{D}})(\mathbb{C})$$

where $\widetilde{G} = \mathrm{SO}(V)$, for the rational quadratic space V with quadratic form defined by $Q(x) = \langle\, x, x \,\rangle$. Thus, $\mathrm{sig}(V) = (2n-2, 2)$.

For a weakly holomorphic form $f \in M^{!,\infty}_{2-n}(D, \chi)$, the Borcherds lift defines a meromorphic modular form $\widetilde{\psi}(f)$ on $\mathrm{Sh}(\widetilde{G}, \widetilde{\mathcal{D}})(\mathbb{C})$.

Let

$$\psi(f) := j^*\widetilde{\psi}(f).$$

Of course, such a Borcherds lift could be defined directly for the dual pair $(\mathrm{U}(1,1), U(n-1,1))$, but it is more efficient to take advantage of the extensively developed theory for the dual pair $(\mathrm{SL}(2), O(2n-2,2))$, via the seesaw.

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Unitary Borcherds forms

Note that the inequivalent cusps of $\Gamma_0(D)$ are $\infty_r \sim \frac{r}{D}$ where $r \mid D$.

For a weakly holomorphic form

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let $\boldsymbol{c}_r(0)$ be its normalized constant term at the cusp ∞_r.
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Note that the inequivalent cusps of $\Gamma_0(D)$ are $\infty_r \sim \frac{r}{D}$ where $r \mid D$.
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Thus, the divisor $\mathrm{div}\,\psi(f)$ involves

(a) the Zariski closure of $\mathcal{Z}^*(m)$ of $\mathcal{Z}(m)$ in $\mathcal{S}^*$

(b) boundary divisors $\mathcal{S}^*(\Phi)$ at the various cusps Φ, with mutiplicities

$$\sum_{m>0} c(-m)\frac{m}{n-2}\left|\{x \in L_\Phi \mid \langle x, x \rangle = m\}\right|$$

(c) components Exc_s of the exceptional locus for the blowup $\mathcal{S}^* = \mathcal{S}^*_{\mathrm{Kra}} \longrightarrow \mathcal{S}^*_{\mathrm{Pap}} \supset \mathrm{Sing}$, with multiplicities

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Green functions

The class of $\mathrm{div}\,\psi(f)$ and of ω^k coincide in the Chow group $\mathrm{CH}^1_{\mathbb{Q}}(\mathcal{S}^*)$, but we still want to include the Green functions.

Following an idea due to Bruinier, consider the space of harmonic Maass forms:

$$H^{\infty}_{2-n}(D,\chi) \supset M^{!,\infty}_{2-n}(D,\chi).$$

These have expansions

$$f(\tau) = \sum_{m \gg -\infty} c^{+}(m)\,\mathbf{q}^m + \sum_{m<0} c^{-}(m)\,\Gamma(n-1, 4\pi|m|v)\,\mathbf{q}^m,$$

where $\tau = u + iv$ and $\Gamma(s,x) = \int_x^\infty e^{-t}\,t^{s-1}\,dt$.
For $m \in \mathbb{Z}_{>0}$, there is a unique such function f_m with

$$f(\tau) = \mathbf{q}^{-m} + O(1), \qquad \text{as } \mathbf{q} \to 0.$$

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One can take such forms as inputs in Borcherds regularized theta integral Θ^{reg}. The crucial facts are:

(1) $\Theta^{\text{reg}}(f_m)$ is a logarithmic Green function[12] on $\mathcal{S}^*(\mathbb{C})$ for the divisor $\mathcal{Z}^{\text{tot}}(m)(\mathbb{C})$.

Therefore we can define

$$\widehat{\mathcal{Z}}^{\text{tot}}(m) = (\mathcal{Z}^{\text{tot}}(m), \Theta^{\text{reg}}(f_m)) \ \in \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*).$$

(2) If $f \in M^{!,\infty}_{2-n}(D, \chi)$ is weakly holomorphic, then

$$\Theta^{\text{reg}}(f) \equiv -\log ||\psi(f)||^2. \quad \text{(up to log} - \text{log-negligible terms)}$$

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Modularity

With this definition of the classes $\widehat{\mathcal{Z}}^{\mathrm{tot}}(m)$ and using (2), we have the relation

$$\widehat{\omega}^k \equiv \widehat{\mathrm{div}}\,\psi(f) := (\mathrm{div}\,\psi(f), -\log||\psi(f)||^2), \qquad \text{in } \widehat{\mathrm{CH}}^1_{\mathbb{Q}}(\mathcal{S}^*).$$

where $k = k(f) = \sum_{r|D} \boldsymbol{c}_r(0)$ depends on f.

Now we do some bookkeeping and use the fact that

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Modularity

By the modularity criterion, it follows that the series

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Computation of the divisor

It remains to explain something about the proof of Theorem A, in particular, about the determination of $\mathrm{div}\psi(f)$ on the integral model $\mathcal{S}^*$

As there are many technical issues, let me just describe the main strategy: *We study $\psi(f)$ and $\widetilde{\psi}(f)$ in a neighborhood of the boundary.*

First consider the complex situation:

$$\begin{array}{ccc} \mathrm{Sh}(G,\mathcal{D})(\mathbb{C}) & \xrightarrow{j} & \mathrm{Sh}(\widetilde{G},\widetilde{\mathcal{D}})(\mathbb{C}) \\ J = \text{isotropic } \Bbbk\text{-line in } V & \Longrightarrow & \text{isotropic } \mathbb{Q}\text{-plane } J \text{ in } V. \\ \text{point boundary component} & \longrightarrow & \text{curve boundary component} \end{array}$$

so that, for the Baily-Borel compactifications

$$\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})^{\mathrm{BB}} \xrightarrow{j} \mathrm{Sh}(\widetilde{G},\widetilde{\mathcal{D}})(\mathbb{C})^{\mathrm{BB}}$$

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In the smooth toroidal compactifications, these are blown up to

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where $\mathcal{KS}_\Phi \longrightarrow \mathcal{Y}_0(D)$ is a Kuga-Sato variety over a modular curve.

Borcherds gave a product formula for $\widetilde{\psi}(f)$ valid in a neighborhood of a point boundary component on $\widetilde{\mathcal{D}}$.

There is another product formula for $\widetilde{\psi}(f)$, valid is a neighborhood of a curve boundary component.

From this product formula, we can read off the Fourier-Jacobi expansion of $\psi(f)$ on the formal completion of $(\mathcal{L}_\Phi^{-1})^\wedge_{\mathcal{B}_\Phi}$,

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Computation of the divisor

In the smooth toroidal compactifications, these are blown up to

$$\left(\mathcal{L}_\Phi^{-1} \longrightarrow \mathcal{B}_\Phi \longrightarrow \mathcal{M}_{(1,0)} \right) \underset{\text{CM-point}}{\longrightarrow} \left(\widetilde{\mathcal{L}}_\Phi^{-1} \longrightarrow \mathcal{KS}_\Phi \longrightarrow \mathcal{Y}_0(D) \right)$$

where $\mathcal{KS}_\Phi \longrightarrow \mathcal{Y}_0(D)$ is a Kuga-Sato variety over a modular curve.
Borcherds gave a product formula for $\widetilde{\psi}(f)$ valid in a neighborhood of a point boundary component on $\widetilde{\mathcal{D}}$.
There is another product formula for $\widetilde{\psi}(f)$, valid is a neighborhood of a curve boundary component.
From this product formula, we can read off the Fourier-Jacobi expansion of $\psi(f)$ on the formal completion of $(\mathcal{L}_\Phi^{-1})^\wedge_{\mathcal{B}_\Phi}$,

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The divisor of $\psi(f)$ on $(\mathcal{L}_\Phi^{-1})^\wedge_{\mathcal{B}_\Phi}$ is then the pullback $\pi^*(\mathrm{div}(\psi_0))$,

$$\begin{tikzcd}
\mathcal{L}_\Phi^{-1} \arrow[r, "\pi"] & \mathcal{B}_\Phi \\
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\end{tikzcd}$$

where ψ_0 is the leading Fourier-Jacobi coefficient.

Note that ψ_0 is a rational section of a certain line bundle $\omega_\Phi^k \cdot \mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)}$ on $\mathcal{B}_\Phi = L_\Phi \otimes E$.

Also, any nonzero vector $x \in L_\Phi$ defines a homomorphism

$$j_x : \mathcal{B}_\Phi = L_\Phi \otimes E \longrightarrow E, \quad \langle x, \cdot \rangle.$$

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Finally, the product formula for $\psi(f)$ shows that

$$\psi_0 = P_\Phi^\eta \cdot P_\Phi^{\text{vert}} \cdot P_\Phi^{\text{hor}}$$

where P_Φ^η is a CM-value of a power of the Dedekind η-function, and

$$P_\Phi^{\text{vert}} = \prod_{r|D} \prod_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b \neq 0 \\ rb = 0}} \Theta(\tau, \frac{b}{D})^{c_r(0)}, \qquad \tau = \text{CM-point}$$

$$P_\Phi^{\text{hor}} = \prod_{m>0} \prod_{\substack{x \in L_\Phi \\ Q(x) = m}} \Theta(\tau, \langle\, w_0, x \,\rangle)^{c(-m)}, \qquad \text{where}$$

$$\Theta(\tau, z) = i \frac{\vartheta_1(\tau, z)}{\eta(\tau)} = q^{\frac{1}{12}} (\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n=1}^{\infty} (1 - \zeta\, q^n)(1 - \zeta^{-1}\, q^n).$$

These are the formulas over $\mathbb{C}$, but the Jacobi theta function lives over $\mathbb{Z}$ and eventually we arrive at Theorem A.

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