# The hyperbolic Ax-Lindemann-Weierstraß conjecture

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Plan of the talk:

(1) Motivation: Hodge locus and André-Oort conjecture.
(2) Bi-algebraic geometry and the Ax-Lindemann-Weierstraß conjecture.
(3) Strategy of the proof.
(4) Some details.

# Motivation: Hodge locus and André-Oort conjecture

- $S$: smooth quasi-projective variety over $\mathbb{C}$.
- $\mathcal{H} \longrightarrow S$ a polarized $\mathbb{Z}$VHS.

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*$HL(\mathcal{H})$ is a countable union of algebraic subvarieties of $S$.*

**Definition:**

*Special subvariety of $(S, \mathcal{H})$:= irreducible stratum of $HL(\mathcal{H})$*
*Special point:= special subvariety of dimension 0*

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Understanding the distribution in $S$ of special subvarieties, especially special points, associated to the $\mathbb{Z}$VHS $\mathcal{H}$.

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**Example**

- $S = SL(2,\mathbb{Z})\backslash\mathbb{H} \overset{j}{\simeq} \mathbb{C} = Y_0(1)$
  moduli space of $\mathbb{C}$-elliptic curves, seen as weight 1 polarized $\mathbb{Z}$HS.
- $\tau \in \mathbb{H} \longleftrightarrow E_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$
- Special point:
  $\tau$ imaginary quadratic $\longleftrightarrow E_\tau$ has complex multiplication (by $\mathbb{Q}(\tau)$)
- Special points are dense in $Y_0(1) = \mathbb{C}$, even for the usual topology.

## Example

- $S = SL(2,\mathbb{Z})\backslash\mathbb{H} \times SL(2,\mathbb{Z})\backslash\mathbb{H} \overset{j}{\simeq} \mathbb{C} \times \mathbb{C} = Y_0(1) \times Y_0(1)$ moduli space of pairs of $\mathbb{C}$-elliptic curves.
- Special points in $Y_0(1) \times Y_0(1)$: pairs $(x, y)$ of special points.
- Special curves:
    - $\{x\} \times Y_0(1)$ or $Y_0(1) \times \{x\}$, $x$ special,
    - $\mathrm{Im}(Y_0(N) \longrightarrow Y_0(1) \times Y_0(1))$, where $Y_0(N)$ is the moduli space of isogenies $\mathbb{Z}/N\mathbb{Z} \hookrightarrow E_1 \twoheadrightarrow E_2$.

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## Conjecture: (André, '89)

*An irreducible curve of $\mathbb{C} \times \mathbb{C}$ containining infinitely many special points is special.*

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**Definition:**

*A connected Shimura variety is a quotient $S = \Gamma\backslash X$ of a symmetric bounded domain $X$ of $\mathbb{C}^N$ by an arithmetic (congruence) subgroup $\Gamma = \mathbf{G}(\mathbb{Z})$, $G = \mathrm{Aut}(X)$.*

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**Example**

- $X = \mathbf{B}^n_{\mathbb{C}}$, $G = PU(n, 1)$.
- $X = D^I_{p,q} = \{Z \in M(p, q, \mathbb{C}) \simeq \mathbb{C}^{pq} : I_q - Z^*Z > 0\}$, $G = PU(p, q)$.
- $X = \{Z \in D^I_{g,g} : Z^t = -Z\}$, $G = Sp(g, \mathbb{R})$, $\Gamma = Sp(g, \mathbb{Z})$, $S = \mathcal{A}_g$ moduli space of Abelian varieties of dimension $g$.

**Basic facts:**

- Such an $S = \Gamma\backslash X$ has a canonical structure of a quasi-projective variety over $\mathbb{C}$ (Baily-Borel), even over $\overline{\mathbb{Q}}$ (Shimura-Deligne).
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The special subvarieties of $S = \Gamma\backslash S$ are completely understood:

- from the group-theoretical point of view: special subvarieties of $S$ are the irreducible components of Hecke translates of Shimura subvarieties of $S$.
- from the differential geometric point of view: special subvarieties are totally geodesic subvarieties containing at least one special point.
- Special points are $\overline{\mathbb{Q}}$-points. They are dense in each special subvariety, in particular in $S$.

### Conjecture: (André-Oort)

*Let $S$ be a connected Shimura variety and $Z \subset S$ a closed irreducible algebraic subvariety.*
*If $Z$ contains a Zariski-dense set of special points then $Z$ is special.*
*Equivalently: there exists finitely many special subvarieties $Y_1, \dots, Y_r$ of $S$ contained in $Z$, and maximal for this property.*

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This conjecture is similar to the Manin-Mumford conjecture:

**Theorem: (Raynaud)**

*Let $A$ be an Abelian variety and $Z \subset A$ a closed irreducible algebraic subvariety.*
*If $Z$ contains a Zariski-dense set of torsion points then $Z$ is a translate of an Abelian subvariety by a torsion point.*

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**Results:**

- The AO conjecture has been proven by Klingler-Ullmo-Yafaev under the Generalized Riemann Hypothesis. The proof uses ergodic, algebraic and arithmetic geometry.
- In 2010 Pila proved unconditionnally the AO conjecture in the special case $S = Y_0(1)^N$. His proof relies on the hyperbolic Ax-Lindemann-Weierstraß conjecture in this particular case.

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- Equivalently: consider the diagram

$$\begin{array}{ccccc} Y & \xrightarrow[\text{maximal}]{\text{algebraic, irreducible}} & \pi^{-1}V & \dashrightarrow & X \\ & & \downarrow & & \downarrow \pi \\ & & V & \xrightarrow{\text{algebraic}} & S. \end{array}$$

Then $Y$ is bialgebraic.

## Example (The flat ALW conjecture)

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## Remark:

This is the geometric analog of the classical Lindemann-Weierstraß theorem:
if $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are $\mathbb{Q}$-linearly independant then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independant over $\mathbb{Q}$.

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Zannier noticed that one can obtain a new proof of the Manin-Mumford conjecture using the Abelian ALW conjecture as a crucial ingredient. Pila realized one might use the same kind of techniques for proving the André-Oort conjecture.

Let

$$\pi : X \longrightarrow S = \Gamma \backslash X$$

be the uniformizing map of an arithmetic variety ($\sim$ connected Shimura variety), where $\Gamma = \mathbf{G}(\mathbb{Z})$ is an arithmetic lattice in $G = \mathrm{Aut}(X)$.

# The hyperbolic ALW conjecture

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**Remark:**

Notice that we are not exactly in the setting of bi-algebraic geometry: while $S$ is an algebraic variety, the bounded symmetric domain $X$ is only a semi-algebraic subset of the algebraic variety $\mathbb{C}^N$.

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**Definition:**

*A subset $Y \subset X$ is an irreducible algebraic subvariety of $X$ if $Y$ is an analytic irreducible component of $D \cap \tilde{Y}$ for $\tilde{Y} \subset \mathbb{C}^N$ an algebraic subvariety.*

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The semi-algebraic structure on $X$ is canonical. If you use any other semi-algebraic realization of $X$ (for example the one given by the Borel embedding) you do not change the algebraic subsets of $X$.

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**Proposition: (Ullmo-Yafaev)**

*The bialgebraic subvarieties in $S$ are the weakly special (i.e. totally geodesic) ones.*

**Theorem: (K., Ullmo, Yafaev)**

*Let* $\pi : X \longrightarrow S = \Gamma\backslash X$ *the uniformizing map of an arithmetic variety. Then the Ax-Lindemann-Weierstraß conjecture holds true for* $\pi$.

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**Corollary:**

*Using this theorem one can obtain a new proof of the André-Oort conjecture under GRH.*

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*The André-Oort conjecture holds true unconditionnally for $S = \mathcal{A}_6^n$, for all $n$.*

# Strategy of the proof of the main theorem:

We start with
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**Theorem (Step 3)**

*The $\mathbb{Q}$-group $H_Y := \left(\overline{\mathbf{G}(\mathbb{Z}) \cap \mathrm{Stab}_{\mathbf{G}(\mathbb{R})} Y}^{\mathrm{Zar}/\mathbb{Q}}\right)^0$ is positive dimensional and stabilizes $Y$.*

Then using classical monodromy arguments (Deligne's semi-simplicity theorem) one can conclude that $Y$ is weakly special.

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To prove Step 3, fix a fundamental set $\mathcal{F} \subset X$ for the $\Gamma$-action.

## Definition:

$\Sigma(Y) := \{g \in \mathbf{G}(\mathbb{R}) \ / \ \dim(gY \cap \pi^{-1}V \cap \mathcal{F}) = \dim Y \}$ .

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Hence we are reduced to showing that $\Sigma(Y)$ contains a positive dimensional semi-algebraic subset.

### Main idea of Pila-Zannier:

even if $\pi : X \longrightarrow S$ is highly transcendental, one can still control its transcendence if it is definable in a "tame topology" in Grothendieck's sense, i.e. an "*o*-minimal structure" in the sense of model theory.

## Main idea of Pila-Zannier:

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## Definition:

*A structure $\mathcal{S}$ is a collection $\mathcal{S} = (S_n)_{n \in \mathbb{N}}$, where $S_n$ is a set of subsets of $\mathbb{R}^n$, called definable sets, such that:*

(1) *all algebraic subsets of $\mathbb{R}^n$ are in $S_n$.*

(2) *$S_n$ is a boolean subalgebra of powerset of $\mathbb{R}^n$.*

(3) *If $A \in S_n$ and $B \in S_m$ then $A \times B \in S_{n+m}$.*

(4) *Let $p : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ be a linear projection. If $A \in S_{n+1}$ the $p(A) \in S_n$.*

*The structure $\mathcal{S}$ is said to be o-minimal if the elements of $S_1$ are precisely the finite unions of points and intervals.*
*A map $f : A \longrightarrow B$ between definable sets is definable if its graph is.*

## Non-trivial *o*-minimal structures do exist:

- $\mathbb{R}_{an}$ (Van den Dries): a function $f : [0,1]^n \longrightarrow \mathbb{R}$ is definable in $\mathbb{R}_{an}$ if it is the restriction of a real analytic function defined on some open neighbourhood of $[0,1]^n$.
- $\mathbb{R}_{exp}$ (Wilkie): you require $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ to be definable.
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- $\mathbb{R}_{\mathrm{an},\exp}$ (Van den Dries-Miller). This is the one used in Diophantine geometry.

### Theorem (Pila-Wilkie)

*Let $Z \subset \mathbb{R}^m$ be definable in some o-minimal structure. Let $Z^{\mathrm{alg}} \subset Z$ be the union of all positive-dimensional semi-algebraic subsets of $Z$. Then:*

$$\forall \varepsilon > 0, \quad \exists C_\varepsilon > 0 \;/\; \left|\left\{x \in (Z \setminus Z^{\mathrm{alg}}) \cap \mathbb{Q}^m), \quad H(x) \leq T\right\}\right| < C_\varepsilon T^\varepsilon \;.$$

### Theorem (Step 1)

*There exists a semi-algebraic fundamental set $\mathcal{F} \subset X$ for $\Gamma$ such that*

$$\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow S$$

*is definable in* $\mathbb{R}_{\mathrm{an,exp}}$.

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$$\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow S$$

*is definable in $\mathbb{R}_{\mathrm{an,exp}}$.*

## Remarks:

- for $S$ compact: this is obvious, even in $\mathbb{R}_{\mathrm{an}}$.
- for $S = \mathcal{A}_g$: this was proven by Peterzil-Starchenko, using explicit intricate computations with $\theta$-functions. Crucially used by Pila-Tsimerman.
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## Corollary:

*The set* $\Sigma(Y) := \{g \in \mathbf{G}(\mathbb{R}) \ / \ \dim(gY \cap \pi^{-1}V \cap \mathcal{F}) = \dim Y \ \}$ *is definable in* $\mathbb{R}_{\text{an,exp}}$.

To show that $\Sigma(Y)$ contains a positive dimensional semi-algebraic set, we are reduced, using the Pila-Wilkie theorem, to showing that

$$\Sigma(Y) \cap \mathbf{G}(\mathbb{Z}) = \{\gamma \in \mathbf{G}(\mathbb{Z}) \;/\; \gamma^{-1}\mathcal{F} \cap Y \neq \emptyset\}$$

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**Theorem: (Step 2)**

*Let $Y \subset X$ be an irreducible algebraic subset of $X$.*
*There exists $c_1 > 0$ such that*

$$|\{\gamma \in \mathbf{G}(\mathbb{Z}) \ / \ Y \cap \gamma\mathcal{F} \neq \emptyset, \ H(\gamma) \leq T\}| \geq T^{c_1} \quad .$$

# About the proof of Step 2 assuming Step 1

**Height on $\mathbf{G}(\mathbb{Z})$:**

Fix $\mathbf{G} \subset \mathbf{GL}(E)$ a faithful linear representation of $\mathbf{G}$. Write $X = \mathbf{G}(\mathbb{R})/K$, where $K$ is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. Fix $\| \cdot \|_\infty$ a $K$-invariant norm on $E$ and denote in the same way the operator norm on $\mathrm{End}\,(E)$.

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**Lemma**

$\exists\, B > 0 \,/\, \forall\, \gamma \in \mathbf{G}(\mathbb{Z}),\ \forall\, u \in \gamma\mathcal{F}, \quad H(\gamma) \leq B\|u\|_\infty.$

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Taking volumes: $\mathrm{Vol}_C(C(T)) \leq \sum_{\substack{\gamma \in \Gamma, \ \gamma\mathcal{F} \cap C \neq \emptyset \\ H(\gamma) \leq B \cdot T}} \mathrm{Vol}_C(\mathcal{F} \cap \gamma^{-1} C) \quad .$

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**Proposition:**

$\exists\, A > 0 \,/\, \forall\, C \subset X$ algebraic curve of degree $d$, $\quad \mathrm{Vol}_C(C \cap \mathcal{F}) \leq A \cdot d \ .$

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**Lemma**

$$\forall\ g \in \mathbf{G}(\mathbb{R}), \quad \log \|g\|_\infty \leq d_X(g \cdot x_0, x_0) \quad .$$

Hence $C \cap B(x_0, \log T) \subset C(T)$. Thus:

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$$|\{\gamma \in \Gamma,\ \gamma\mathcal{F} \cap C \neq \emptyset,\ H(\gamma) \leq B \cdot T\}| \geq \frac{1}{A \cdot d} \mathrm{Vol}_C(C \cap B(x_0, \log T))$$

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# About the proof of Theorem 1.

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Let us consider the diagram of holomorphic maps:

$$\mathcal{F} \subset \mathbb{H} \overset{z \mapsto e^{2\pi i z}}{\longrightarrow} D^* \overset{j}{\longrightarrow} S = \mathbb{C} \ ,$$

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We claim that this composite is definable in $\mathbb{R}_{\text{an,exp}}$. It follows from the following observations:

- $\exp(2\pi i z) = \exp(-2\pi \mathrm{Im}(z)) \cdot \exp(2\pi i \mathrm{Re}(z))$. The first factor is definable is definable in $\mathbb{R}_{\text{exp}}$. On the other hand $\mathrm{Re}(x)$ is bounded on $\mathcal{F}$, hence the second factor, on $\mathcal{F}$, is definable in $\mathbb{R}_{\text{an}}$.
- The $j$-function $j : D^* \longrightarrow \mathbb{C}$ extends to $D \longrightarrow \mathbf{P}^1\mathbb{C}$ hence is definable in $\mathbb{R}_{\text{an}}$.

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This picture extends to any arithmetic variety using toroïdal compactifications for $S$ and Siegel fundamental domain in $X$ for $\Gamma$.