# Twists of Elliptic curves

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September 23, 2016

$K$, a number field

$E/K$, an elliptic curve over $K$, the one given by the equation

$$y^2 = x^3 + ax + b \quad (a, b \in K).$$

## Definition

A twist of $E/K$ is another elliptic curve which is isomorphic to $E$ over $\bar{K}$.

We focus on quadratic twists and cubic twists:

- The quadratic twist of $E$ by square-free $D \in K$ is :

$$\begin{array}{ccc} E : y^2 = x^3 + ax + b & \simeq & E^D : y^2 = x^3 + aD^2x + bD^3, \\ (x, y) & \mapsto & (\frac{x}{D}, \frac{y}{D\sqrt{D}}) \end{array}$$

- The cubic twist of $E$ by cube-free $D \in K$ is :

$$\begin{array}{ccc} E : y^2 = x^3 + b & \simeq & E_D : y^2 = x^3 + bD^2, \\ (x, y) & \mapsto & (\frac{x}{\sqrt[3]{D^2}}, \frac{y}{D}) \end{array}$$

# Quadratic twists

Let $K = \mathbb{Q}$. The root number of $E/\mathbb{Q}$ is

$$\begin{cases} +1, & \text{if } E \text{ has even analytic rank,} \\ -1, & \text{if } E \text{ has odd analytic rank.} \end{cases}$$

Define

$$S(X) := \{\text{square-free } D \in \mathbb{Z} : |D| \leq X\}.$$

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It is known that

$$\lim_{X \to \infty} \frac{\#\{D \in S(X) : \text{the root number of } E^D/\mathbb{Q} \text{ is } 1\}}{\#S(X)} = \frac{1}{2}.$$

The Parity Conjecture is a weak form of the BSD conjecture.
It says that the rank and the analytic rank of $E$ have the same parity.

Assuming the Parity Conjecture, the above says that

$$\lim_{X \to \infty} \frac{\#\{D \in S(X) : \text{the rank of } E^D/\mathbb{Q} \text{ is even}\}}{\#S(X)} = \frac{1}{2}.$$

Thus the average rank of quadratic twists is at least $\frac{1}{2}$.

# Quadratic twists : Goldfeld's conjecture

If $E$ is an elliptic curve over $\mathbb{Q}$, then

## Goldfeld's Conjecture, 1979

$$\lim_{X \to \infty} \frac{\sum_{D \in S(X)} \mathrm{rank}(E^D(\mathbb{Q}))}{\#S(X)} = \frac{1}{2}.$$

Assuming the Parity conjecture, Goldfeld's conjecture asserts that

$$\begin{aligned}
\mathrm{rank}\, E^D/\mathbb{Q} &= 0 \quad \text{for } 50\% \text{ square-free } D\text{'s},\\
\mathrm{rank}\, E^D/\mathbb{Q} &= 1 \quad \text{for } 50\% \text{ square-free } D\text{'s},\\
\mathrm{rank}\, E^D/\mathbb{Q} &\geq 2 \quad \text{for } \ 0\% \text{ square-free } D\text{'s}.
\end{aligned}$$

# Cubic twists

Consider the curves

$$E_m : x^3 + y^3 = D \quad (\cong y^2 = x^3 - 2^4 3^3 D^2).$$

These are cubic twists of $E : x^3 + y^3 = 1$. Let

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## Zagier and Kramarz's Conjecture, 1987

$$\frac{\#\{D \in C(X) : \text{analytic rank of } E_D/\mathbb{Q} \geq 2\}}{\# C(X)} > 0 \quad \text{as } X \to \infty.$$

# Over number fields - Quadratic twists

## Example

Let $K = \mathbb{Q}(i)$ and $E/K : y^2 = x^3 + x$. This is a CM curve,

$$\mathrm{End}_K E \cong \mathbb{Z}[i], \quad [i](x,y) = (-x, iy).$$

Thus every quadratic twists of $E/K$ has even rank.

## Theorem (Dokchitser-Dokchitser, 2009)

*The root number of $E^D/K$ is the root number of $E/K$ for all $D$ iff*

1. *$K$ is totally complex, and*
2. *For all primes $\mathfrak{p}$ of $K$ the curve $E/K_{\mathfrak{p}}$ acquires good reduction over an abelian extension of $K_{\mathfrak{p}}$.*

# Over number fields - Cubic twists

Let $E/K$ be the elliptie curve $y^2 = x^3 + b$.

## Theorem (Byeon-K.)

*The root number of $E_D/K$ is the root number of $E/K$ for all $D$ iff $K$ contains the primitive cube root of unity.*