Tropical Currents

June Huh

September 30, 2014

The *logarithm map* is the homomorphism

$$\text{Log} : \mathbb{C}^* \longrightarrow \mathbb{R}, \qquad z \longmapsto -\log|z|,$$

and the *argument map* is the homomorphism

$$\text{Arg} : \mathbb{C}^* \longrightarrow S^1, \qquad z \longmapsto z/|z|.$$

Let N be a finitely generated free abelian group.

There are homomorphisms

$$\begin{tikzcd} & T_N \arrow[dl, "\mathrm{Log} \otimes_{\mathbb{Z}} 1"'] \arrow[dr, "\mathrm{Arg} \otimes_{\mathbb{Z}} 1"] & \\ N_{\mathbb{R}} & & S_N, \end{tikzcd}$$

called the *logarithm map* and the *argument map* for N respectively, where

$$\begin{aligned} T_N &:= \text{the complex algebraic torus } \mathbb{C}^* \otimes_{\mathbb{Z}} N, \\ S_N &:= \text{the compact real torus } S^1 \otimes_{\mathbb{Z}} N, \\ N_{\mathbb{R}} &:= \text{the real vector space } \mathbb{R} \otimes_{\mathbb{Z}} N. \end{aligned}$$

Corresponding to a p-dimensional rational subspace H of $\mathbb{R}^n$, there is

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_{H\cap\mathbb{Z}^n} \arrow[r] \arrow[d] & (S^1)^n \arrow[r] \arrow[d, bend right] & S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{H\cap\mathbb{Z}^n} \arrow[r] \arrow[d] & (\mathbb{C}^*)^n \arrow[r] \arrow[d, "\text{Log}"] \arrow[u, bend right, "\text{Arg}"'] & T_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[d] & \mathbb{R}^n \arrow[r] \arrow[d] & \mathbb{R}^n/H \arrow[r] \arrow[d] & 0, \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the vertical surjections are the logarithm maps.

We define a homomorphism π_H as the composition

$$\pi_H : \ \mathrm{Log}^{-1}(H) \xrightarrow{\mathrm{Arg}} (S^1)^n \longrightarrow S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)} \ .$$

Its kernel is the closed subgroup

$$\ker(\pi_H) = T_{H\cap\mathbb{Z}^n},$$

and all other fibers are translations of the kernel by the action of $(S^1)^n$.

Let A be a p-dimensional affine subspace of $\mathbb{R}^n$ parallel to the linear subspace H.

We define a submersion π_A as the composition

$$\pi_A : \ \mathrm{Log}^{-1}(A) \xrightarrow{e^a} \mathrm{Log}^{-1}(H) \xrightarrow{\pi_H} S_{\mathbb{Z}^n/(H\cap\mathbb{Z}^n)}, \qquad a \in A.$$

The submersion π_A does not depend on the choice of a.

Let σ be a p-dimensional rational polyhedron in $\mathbb{R}^n$. We define

$$\begin{aligned} \text{aff}(\sigma) &:= \text{the affine span of } \sigma, \\ \sigma^\circ &:= \text{the interior of } \sigma \text{ in aff}(\sigma), \\ H_\sigma &:= \text{the linear subspace parallel to aff}(\sigma). \end{aligned}$$

The *normal lattice* of σ is the quotient group

$$N(\sigma) := \mathbb{Z}^n/(H_\sigma \cap \mathbb{Z}^n).$$

The normal lattice defines the $(n-p)$-dimensional vector spaces

$$N(\sigma)_\mathbb{R} := \mathbb{R} \otimes_\mathbb{Z} N(\sigma), \qquad N(\sigma)_\mathbb{C} := \mathbb{C} \otimes_\mathbb{Z} N(\sigma).$$

We define a submersion π_σ as the restriction of $\pi_{\mathrm{aff}(\sigma)}$ to $\mathrm{Log}^{-1}(\sigma^\circ)$:

$$\pi_\sigma : \mathrm{Log}^{-1}(\sigma^\circ) \longrightarrow S_{N(\sigma)}.$$

Each fiber $\pi_\sigma^{-1}(x)$ is a p-dimensional complex submanifold of $(\mathbb{C}^*)^n$.

We define a submersion π_σ as the restriction of $\pi_{\text{aff}(\sigma)}$ to $\text{Log}^{-1}(\sigma^\circ)$:

$$\pi_\sigma : \text{Log}^{-1}(\sigma^\circ) \longrightarrow S_{N(\sigma)}.$$

Each fiber $\pi_\sigma^{-1}(x)$ is a p-dimensional complex submanifold of $(\mathbb{C}^*)^n$.

Definition

Let μ be a complex Borel measure on $S_{N(\sigma)}$. We define a (p, p)-dimensional current

$$\mathscr{T}_\sigma(\mu) := \int_{x \in S_{N(\sigma)}} \left[\pi_\sigma^{-1}(x)\right] d\mu(x).$$

If μ is the normalized Haar measure, we write

$$\mathscr{T}_\sigma := \mathscr{T}_\sigma(\mu).$$

A p-dimensional *weighted complex* in $\mathbb{R}^n$ is a polyhedral complex $\mathscr{C}$ such that

1. each inclusion-maximal cell σ in $\mathscr{C}$ is rational,
2. each inclusion-maximal cell σ in $\mathscr{C}$ is p-dimensional, and
3. each inclusion-maximal cell σ in $\mathscr{C}$ is assigned a complex number $w_{\mathscr{C}}(\sigma)$.

A p-dimensional *weighted complex* in $\mathbb{R}^n$ is a polyhedral complex $\mathscr{C}$ such that

1. each inclusion-maximal cell σ in $\mathscr{C}$ is rational,
2. each inclusion-maximal cell σ in $\mathscr{C}$ is p-dimensional, and
3. each inclusion-maximal cell σ in $\mathscr{C}$ is assigned a complex number $w_{\mathscr{C}}(\sigma)$.

The weighted complex $\mathscr{C}$ is said to be *positive* if, for all p-dimensional cells σ in $\mathscr{C}$,

$$w_{\mathscr{C}}(\sigma) \geq 0.$$

The *support* of $\mathscr{C}$ is the union of all p-dimensional cells of $\mathscr{C}$ with nonzero weight.

Definition

We define a (p,p)-dimensional current on $(\mathbb{C}^*)^n$ by

$$\mathscr{T}_{\mathscr{C}} := \sum_{\sigma} w_{\mathscr{C}}(\sigma)\ \mathscr{T}_{\sigma},$$

where the sum is over all p-dimensional cells in $\mathscr{C}$.

Let τ be a codimension 1 face of a p-dimensional rational polyhedron σ.

The difference of σ and τ defines a ray in the normal space

$$\text{cone}(\sigma - \tau)/H_\tau \subseteq H_\sigma/H_\tau \subseteq \mathbb{R}^n/H_\tau = N(\tau)_\mathbb{R}.$$

We write $u_{\sigma/\tau}$ for the primitive generator of this ray in the lattice $N(\tau)$.

Let τ be a codimension 1 face of a p-dimensional rational polyhedron σ.

The difference of σ and τ defines a ray in the normal space

$$\text{cone}(\sigma - \tau)/H_\tau \subseteq H_\sigma/H_\tau \subseteq \mathbb{R}^n/H_\tau = N(\tau)_\mathbb{R}.$$

We write $u_{\sigma/\tau}$ for the primitive generator of this ray in the lattice $N(\tau)$.

Definition

A p-dimensional weighted complex $\mathscr{C}$ satisfies the *balancing condition* at τ if

$$\sum_{\sigma \supset \tau} w_\mathscr{C}(\sigma) u_{\sigma/\tau} = 0$$

in $N(\tau)_\mathbb{C}$, where the sum is over all p-dimensional cells σ in $\mathscr{C}$ containing τ.

A weighted complex is *balanced* if it satisfies the balancing condition at each of its codimension 1 cells.

A *tropical variety* is a positive and balanced weighted complex, and a *tropical current* is the current associated to a tropical variety.

Theorem (A)

A weighted complex $\mathscr{C}$ is balanced if and only if $\mathscr{T}_{\mathscr{C}}$ is closed.

Definition

A normal closed current $\mathscr{T}$ is *strongly extremal* if for any normal closed current $\mathscr{T}'$ which has the same dimension and support as $\mathscr{T}$ there is a complex number c such that $\mathscr{T}' = c \cdot \mathscr{T}$.

Definition

A normal closed current $\mathscr{T}$ is *strongly extremal* if for any normal closed current $\mathscr{T}'$ which has the same dimension and support as $\mathscr{T}$ there is a complex number c such that $\mathscr{T}' = c \cdot \mathscr{T}$.

If $\mathscr{T}$ is positive and strongly extremal, then $\mathscr{T}$ generates an extremal ray in the cone of positive closed currents: If $\mathscr{T} = \mathscr{T}_1 + \mathscr{T}_2$ is any decomposition of $\mathscr{T}$ into positive closed currents, then

$$|\mathscr{T}| = |\mathscr{T} + \mathscr{T}_1| = |\mathscr{T} + \mathscr{T}_2|,$$

and hence both $\mathscr{T}_1$ and $\mathscr{T}_2$ are positive multiples of $\mathscr{T}$.

Definition

A balanced complex $\mathscr{C}$ is *strongly extremal* if for any balanced complex $\mathscr{C}'$ which has the same dimension and support as $\mathscr{C}$ there is a complex number c such that $\mathscr{C}' \sim c \cdot \mathscr{C}$.

A weighted complex in $\mathbb{R}^n$ is *non-degenerate* if its support is contained in no proper affine subspace of $\mathbb{R}^n$.

Theorem (B)

A non-degenerate weighted complex $\mathscr{C}$ is strongly extremal if and only if $\mathscr{T}_{\mathscr{C}}$ is strongly extremal.

Let $\mathscr{P}$ be a p-dimensional locally finite polyhedral complex in $\mathbb{R}^n$.

Choose a complex Borel measure μ_σ for each p-dimensional cell σ of $\mathscr{P}$, and define

$$\mathscr{T} := \sum_{\sigma} \mathscr{T}_\sigma(\mu_\sigma).$$

For each σ and its codimension 1 face τ, there are inclusion maps

$$M(\sigma) \longrightarrow M(\tau) \longrightarrow (\mathbb{Z}^n)^\vee$$

dual to the quotient maps

$$\mathbb{Z}^n \longrightarrow N(\tau) \longrightarrow N(\sigma).$$

If m is an element of $M(\sigma)$, the *m-th Fourier coefficient* of μ_σ is the complex number

$$\hat{\mu}_\sigma(m) := \int_{x \in S_N} \chi^m \, d\mu_\sigma(x)$$

For a p-dimensional cell σ in $\mathscr{P}$ and an element m of $(\mathbb{Z}^n)^\vee$, we set

$$w_{\mathscr{T}}(\sigma, m) := \begin{cases} \hat{\mu}_\sigma(m) & \text{if } m \in M(\sigma), \\ 0 & \text{if } m \notin M(\sigma). \end{cases}$$

This defines a p-dimensional weighted complex $\mathscr{C}_{\mathscr{T}}(m)$ in $\mathbb{R}^n$.

For a p-dimensional cell σ in $\mathscr{P}$ and an element m of $(\mathbb{Z}^n)^\vee$, we set

$$w_{\mathscr{T}}(\sigma, m) := \begin{cases} \hat{\mu}_\sigma(m) & \text{if } m \in M(\sigma), \\ 0 & \text{if } m \notin M(\sigma). \end{cases}$$

This defines a p-dimensional weighted complex $\mathscr{C}_{\mathscr{T}}(m)$ in $\mathbb{R}^n$.

Theorem (C)

The current $\mathscr{T}$ is closed if and only if $\mathscr{C}_{\mathscr{T}}(m)$ is balanced for all $m \in (\mathbb{Z}^n)^\vee$.

For a p-dimensional cell σ in $\mathscr{P}$ and an element m of $(\mathbb{Z}^n)^\vee$, we set

$$w_{\mathscr{T}}(\sigma, m) := \begin{cases} \hat{\mu}_\sigma(m) & \text{if } m \in M(\sigma), \\ 0 & \text{if } m \notin M(\sigma). \end{cases}$$

This defines a p-dimensional weighted complex $\mathscr{C}_{\mathscr{T}}(m)$ in $\mathbb{R}^n$.

Theorem (C)

The current $\mathscr{T}$ is closed if and only if $\mathscr{C}_{\mathscr{T}}(m)$ is balanced for all $m \in (\mathbb{Z}^n)^\vee$.

Theorem (C)

The current $\mathscr{T}$ is closed if and only if $\mathscr{C}_{\mathscr{T}}(m)$ is balanced for all $m \in (\mathbb{Z}^n)^\vee$.

This implies Theorem (A) because $\hat{\mu}(m) = 0$ for $m \neq 0$ if μ is an invariant measure.

This implies Theorem (B) because $\bigcap_\sigma M(\sigma) = 0$ if $\mathscr{P}$ is non-degenerate.

Let X be the n-dimensional smooth complex toric variety of a complete fan Σ.

A cohomology class in X gives a homomorphism from the homology group of X to $\mathbb{Z}$, defining the Kronecker duality homomorphism

$$H^{2k}(X;\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(H_{2k}(X;\mathbb{Z}),\mathbb{Z}\big).$$

The above homomorphism is, in fact, an isomorphism.

Let X be the n-dimensional smooth complex toric variety of a complete fan Σ.

A cohomology class in X gives a homomorphism from the homology group of X to $\mathbb{Z}$, defining the Kronecker duality homomorphism

$$H^{2k}(X;\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(H_{2k}(X;\mathbb{Z}),\mathbb{Z}\big).$$

The above homomorphism is, in fact, an isomorphism.

Since the homology group is generated by k-dimensional torus orbit closures, the duality identifies cohomology classes with certain integer valued functions on the set of p-dimensional cones in Σ, where $p = n - k$.

Theorem (Fulton-Sturmfels)

The Kronecker duality gives isomorphisms between abelian groups

$$H^{2k}(X;\mathbb{Z}) \simeq \textit{Hom}\big(H_{2k}(X), \mathbb{Z}\big) \simeq \big\{p\text{-dimensional balanced integral weights on } \Sigma\big\},$$

There is an induced isomorphism between complex vector spaces

$$H^{k,k}(X) \simeq \big\{p\text{-dimensional balanced weights on } \Sigma\big\}.$$

The *recession cone* of a polyhedron σ is the convex polyhedral cone

$$\mathrm{rec}(\sigma) = \{b \in \mathbb{R}^n \mid \sigma + b \subseteq \sigma\}.$$

Definition

Let $\mathscr{C}$ be a p-dimensional finite weighted complex in $\mathbb{R}^n$.

For each p-dimensional cone γ in Σ, we define

$$w_{\mathrm{rec}(\mathscr{C},\Sigma)}(\gamma) := \sum_{\sigma} w_{\mathscr{C}}(\sigma),$$

where the sum is over all p-dimensional cells σ in $\mathscr{C}$ whose recession cone is γ.

This defines a p-dimensional weighted complex $\mathrm{rec}(\mathscr{C}, \Sigma)$, the *recession* of $\mathscr{C}$ in Σ.

We say that $\mathscr{C}$ is *compatible* with Σ if $\text{rec}(\sigma) \in \Sigma$ for all $\sigma \in \mathscr{C}$. In this case, we write

$$\text{rec}(\mathscr{C}) := \text{rec}(\mathscr{C}, \Sigma).$$

For any $\mathscr{C}$, there is a subdivision of $\mathscr{C}$ that is compatible with a subdivision of Σ.

Theorem (D)

If $\mathscr{C}$ is a p-dimensional balanced weighted complex compatible with Σ, then

$$\{\overline{\mathscr{T}}_{\mathscr{C}}\} = \textit{rec}(\mathscr{C}) \in H^{k,k}(X).$$

In particular, if all polyhedrons in $\mathscr{C}$ are cones in Σ, then

$$\{\overline{\mathscr{T}}_{\mathscr{C}}\} = \mathscr{C} \in H^{k,k}(X).$$

Theorem (E)

There is a $(2,2)$-dimensional positive closed current T on a smooth projective variety X of dimension 4 such that

1. $\{T\} \in H^{2,2}(X) \cap H^4(X;\mathbb{Z})$, *and*
2. T *is not a limit of currents of the form* $\sum \lambda_i [V_i]$ *with* $\lambda_i > 0$.

The current T is strongly extremal, and this property is crucial in justifying 2.

If T is positive and strongly extremal, then T spans an extremal ray of the cone of positive closed currents.

Therefore, if T is a limit of currents of the form $\sum_i \lambda_i[V_i]$,$\lambda_i > 0$ then T is a limit of currents of the form $\lambda[V]$, $\lambda > 0$.

Lemma

$$L := \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 3 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 3 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}.$$

The characteristic polynomial of L is

$$\chi_L(x) = x^4(x^3 - 4x^2 - x + 8)^2(x^4 - 8x^3 + 12x^2 + 24x - 35).$$