## Dynamics of energy critical wave equations

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We are interested in the dynamics of nonlinear wave equations.

Before considering nonlinear waves, let us begin with linear waves:

$$\partial_{tt}u - \Delta u = 0, \quad (x,t) \in R^d \times R.$$

For the linear wave, we have plane wave solutions: for each  $\xi \in R^d$ ,  $e^{i(\xi \cdot x + |\xi|t)}$  is a solution. These solutions move with speed one in the direction of  $\frac{\xi}{|\xi|}$ .

By linearity,  $\int_{R^d} f(\xi) e^{i(\xi \cdot x + |\xi|t)} d\xi$  is also a solution. By simple integration by parts, this solution is supported in the region  $\{x : \left|\frac{\xi}{|\xi|}t - x\right| = O(1), \text{ for some } \xi\} \subset \{x : ||x| - t| = O(1)\}.$ 

Thus, at large times t, the solution is spread out on an annulus region of radius t and thickness O(1). This is the decay mechanism of the linear waves. Dispersion leads to decay.

The situation for nonlinear waves could be very different. Now one can have "solitons" which do not disperse. In this case, it becomes much more complicated how to capture the dispersive property of the linear waves.

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For some integrable equations (such as the historically important KdV equation modelling water waves in a channel), one can show that solutions will eventually break up as solitons plus radiation ("Soliton resolution") using techniques specific to integrable equations. These techniques do not apply in the general case.

There are many (non-integrable) models for soliton dynamics. In this talk, we will focus on the following focusing energy critical wave equation

$$\partial_{tt} u - \Delta u = u^5, \tag{1}$$

in  $R^3 \times [0,\infty)$ .

An important feature for the focusing energy critical wave equation

$$\partial_{tt}u - \Delta u = u^5$$

is that there are infinitely many steady states Q:

$$-\Delta Q = Q^5.$$

The equation is invariant under translations, scaling

$$u(x,t) \rightarrow u_{\lambda}(x,t) = \lambda^{\frac{1}{2}} u(\lambda x, \lambda t),$$

and Lorentz transformations:

$$u(x,t) \rightarrow u_{\ell}(x,t) = u\left(x - \frac{x \cdot \ell}{|\ell|} \frac{\ell}{|\ell|} + \frac{\frac{x \cdot \ell}{|\ell|} \frac{\ell}{|\ell|} - \ell t}{\sqrt{1 - |\ell|^2}}, \frac{t - x \cdot \ell}{\sqrt{1 - |\ell|^2}}\right),$$

for each  $\ell \in \mathbb{R}^n$  with  $|\ell| < 1$ .

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Combined with the steady states, we obtain traveling wave solutions:

$$\mathcal{Q}_\ell(x,t) = \mathcal{Q}\left(x - rac{x \cdot \ell}{|\ell|} \, rac{\ell}{|\ell|} + rac{rac{x \cdot \ell}{|\ell|} \, rac{\ell}{|\ell|} - \ell t}{\sqrt{1 - |\ell|^2}}
ight),$$

where Q is a steady state and  $|\ell| < 1$ .  $Q_{\ell}$  travels in the direction of  $\ell$  with speed  $|\ell|$ .

We will consider type II solutions only, i.e., solution u with

$$\sup_{t\in(0,T_{\max})}\|\overrightarrow{u}\|_{\dot{H}^1\times L^2}<\infty.$$

This condition is used to rule out ODE type behavior.

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There are rich dynamics for type II solutions:

1. Small solutions are type II, and they scatter, i.e.,  $\exists$  linear solution  $\overrightarrow{u}^L$  s.t.

$$\lim_{t\to\infty} \|\overrightarrow{u}(t)-\overrightarrow{u}^{L}(t)\|_{\dot{H}^1\times L^2}=0.$$

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- 2. Travelling waves are type II, but do not scatter.
- 3. There are type II blow up solutions:

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$$u(x,t) = \lambda(t)^{-\frac{1}{2}}W\left(\frac{x}{\lambda(t)}\right) + \epsilon(x,t),$$

where  $\lambda(t) = (T_+ - t)^{1+\nu} \rightarrow 0$  as  $t \rightarrow T_+$  and  $\epsilon(t)$  is small in energy space (Krieger, Schlag and Tataru; Hillairet and Raphael).

### Soliton resolution Conjecture for type II solutions

In this context, soliton resolution conjecture can be formulated as: Any type II solution  $\overrightarrow{u}(t)$  can be decomposed as a finite sum of modulated solitons, a linear wave, plus a term which vanishes asymptotically in the energy space as  $t \to T_+$ .

## Soliton resolution conjecture for type II solutions



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### Soliton resolution conjecture

More preciely, the conjecture predicts that

$$egin{aligned} \overrightarrow{u}(t) &=& \sum_{j=1}^J \left(\lambda_j(t)^{-rac{1}{2}} \mathcal{Q}_{\ell_j}\left(rac{x-x_j(t)}{\lambda_j(t)}
ight),\,\lambda_j(t)^{-rac{3}{2}} \partial_t \mathcal{Q}_{\ell_j}\left(rac{x-x_j(t)}{\lambda_j(t)}
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where  $u^L$  is a linear wave, and  $(\epsilon(t), \partial_t \epsilon(t)) = o_{\dot{H}^1 \times L^2}(1)$  as  $t \to T_+$ .

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The soliton resolution conjecture was settled in a remarkable work of Duyckaerts, Kenig and Merle, in the radial case for d = 3. The nonradial case is still open.

A crucial new tool introduced by Duyckaerts, Kenig and Merle, in the proof of soliton resolution conjecture, is the channel of energy inequality: for finite energy radial solution  $u^L$  to the linear wave equation  $\partial_{tt} u^L - \Delta u^L = 0$  in  $R^d \times R$ , we have

$$\int_{|x|>R+|t|} \frac{|\nabla_{x,t} u^L|^2}{2}(x,t) \, dx \ge c_0 \int_{R^3} \frac{|\nabla_{x,t} u^L|^2}{2}(x,t) \, dx$$

for all  $t \ge 0$  or all  $t \le 0$ , and some  $R \ge 0$ , under various restrictions on  $(u_0, u_1)$  depending on the dimension d.

A crucial point of the channel of energy inequality is that it implies for radial finite energy solutions to the linear wave equation, a fixed portion of the energy moves out with speed exactly equal to 1, a sharp counterpart of the finite speed of propagation.

The channel of energy inequality implies that if a radial finite energy solution u is not a steady state, then

$$\int_{|x|>R+|t|} |\nabla_{x,t}u|^2(x,t) \, dx > \delta > 0$$

for some R > 0 and all  $t \ge 0$  or all  $t \le 0$ .

Now, heuristically at least, the soliton resolution should hold: if the solution does not end in the form of finite sum of rescaled solitons, then it must emit energy.

This process can not repeat itself forever, due to energy constraints.

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The non-radial case is much less understood. We only have a recent partial result.

### Soliton resolution along a sequence of times, nonradial case

#### We have

#### Theorem (J., 2015, Duyckaerts, J., Kenig and Merle 2016)

Let  $\overrightarrow{u}$  be a Type II blow up solution. Define the singular set

$$S := \left\{ x_* \in \mathbb{R}^d : \|u\|_{\substack{\frac{d+2}{d-2} \\ L_t^{\frac{d-2}{d-2}}} L_x^2 \frac{d+2}{d-2}} (B_{\epsilon}(x_*) \times [\mathcal{T}_+ - \epsilon, \mathcal{T}_+))} = \infty, \quad \text{for any } \epsilon > 0 \right\}.$$
(2)

Then S is a set of finitely many points only. Then near a singular point, we have

$$\vec{u}(t_n) = \vec{v} + \sum_{j=1}^{J_*} \left( (\lambda_n^j)^{-\frac{d}{2}+1} \, \mathcal{Q}_{\ell_j}\left( \frac{x - c_n^j}{\lambda_n^j} \,, \, 0 \right) \,, \, (\lambda_n^j)^{-\frac{d}{2}} \, \partial_t \, \mathcal{Q}_{\ell_j}\left( \frac{x - c_n^j}{\lambda_n^j} \,, \, 0 \right) \right) + o_{\dot{H}^1 \times L^2}(1), \tag{3}$$

as  $n \to \infty$ .

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There is a corresponding version for global Type II solutions.

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## Soliton resolution along a sequence of times, singular case



### Elimination of dispersive energy, illustrated



# Thank you!